Measuring Angles and Arcs

Why?

The thirteen stars of the Betsy Ross flag are arranged equidistant from each other and from a fixed point. The distance between consecutive stars varies depending on the size of the flag, but the measure of the central angle formed by the center of the circle and any two consecutive stars is always the same.

Angles and Arcs

A central angle of a circle is an angle with a vertex in the center of the circle. Its sides contain two radii of the circle. \( \angle ABC \) is a central angle of \( \odot B \).

Recall from Lesson 1-4 that a degree is \( \frac{1}{360} \) of the circular rotation about a point. This leads to the following relationship.

Key Concept

**Sum of Central Angles**

**Words** The sum of the measures of the central angles of a circle with no interior points in common is 360.

**Example** \( m \angle 1 + m \angle 2 + m \angle 3 = 360 \)

![Example Image]

**EXAMPLE 1** Find Measures of Central Angles

Find the value of \( x \).

\[
m \angle GFH + m \angle HFJ + m \angle GFJ = 360 \]

\[
130 + 90 + m \angle GFJ = 360 \]

\[
220 + m \angle GFJ = 360 \]

\[
m \angle GFJ = 140
\]

**Check Your Progress**

1A. \( 145^\circ \) \( 165^\circ \) \( x^\circ \)

1B. \( 40^\circ \) \( 85^\circ \) \( x^\circ \)

An arc is a portion of a circle defined by two endpoints. A central angle separates the circle into two arcs with measures related to the measure of the central angle.
**Lesson 10-2**

**Measuring Angles and Arcs**

**Key Concept**

<table>
<thead>
<tr>
<th>Arc</th>
<th>Measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>A <strong>minor arc</strong> is the shortest arc connecting two endpoints on a circle.</td>
<td>The measure of a minor arc is less than 180 and equal to the measure of its related central angle. $m\widehat{AB} = m\angle ACB = x$</td>
</tr>
<tr>
<td>A <strong>major arc</strong> is the longest arc connecting two endpoints on a circle.</td>
<td>The measure of a major arc is greater than 180, and equal to 360 minus the measure of the minor arc with the same endpoints. $m\widehat{ADB} = 360 - m\widehat{AB} = 360 - x$</td>
</tr>
<tr>
<td>A <strong>semicircle</strong> is an arc with endpoints that lie on a diameter.</td>
<td>The measure of a semicircle is 180. $m\widehat{ADB} = 180$</td>
</tr>
</tbody>
</table>

**StudyTip**

**Naming Arcs** Minor arcs are named by their endpoints. Major arcs and semicircles are named by their endpoints and another point on the arc that lies between these endpoints.

**EXAMPLE 1**

**Classify Arcs and Find Arc Measures**

\(\widehat{GJ}\) is a diameter of \(\odot K\). Identify each arc as a major arc, minor arc, or semicircle. Then find its measure.

- **a.** \(m\widehat{GH}\)
  
  \(\widehat{GH}\) is a minor arc, so \(m\widehat{GH} = m\angle GKH\) or 122.

- **b.** \(m\widehat{GLH}\)
  
  \(\widehat{GLH}\) is a major arc that shares the same endpoints as minor arc \(\widehat{GH}\).  
  
  \(m\widehat{GLH} = 360 - m\widehat{GH}\)  
  
  \(= 360 - 122\) or 238

**EXAMPLE 2**

**Classify Arcs and Find Arc Measures**

\(\widehat{GJ}\) is a diameter of \(\odot K\). Identify each arc as a major arc, minor arc, or semicircle. Then find its measure.

- **a.** \(m\widehat{GH}\)
  
  \(\widehat{GH}\) is a minor arc, so \(m\widehat{GH} = m\angle GKH\) or 122.

- **b.** \(m\widehat{GLH}\)
  
  \(\widehat{GLH}\) is a major arc that shares the same endpoints as minor arc \(\widehat{GH}\).  
  
  \(m\widehat{GLH} = 360 - m\widehat{GH}\)  
  
  \(= 360 - 122\) or 238

**Check Your Progress**

\(\overline{PM}\) is a diameter of \(\odot R\). Identify each arc as a major arc, minor arc, or semicircle. Then find its measure.

- **2A.** \(\widehat{MQ}\)
  
  \(\widehat{MQ}\) is a semicircle, so \(m\widehat{MQ} = 180\).

- **2B.** \(\widehat{MNP}\)
  
  \(\widehat{MNP}\) is a minor arc, so \(m\widehat{MNP} = m\angle MNP\) or 115.

- **2C.** \(\widehat{MNQ}\)
  
  \(\widehat{MNQ}\) is a major arc that shares the same endpoints as minor arc \(\widehat{MQ}\).  
  
  \(m\widehat{MNQ} = 360 - m\widehat{MQ}\)  
  
  \(= 360 - 180\) or 180

**Congruent arcs** are arcs in the same or congruent circles that have the same measure.

**Theorem 10.1**

**Words** In the same circle or in congruent circles, two minor arcs are congruent if and only if their central angles are congruent.

**Example** If \(\angle 1 \cong \angle 2\), then \(\widehat{FG} \cong \widehat{HJ}\).  

If \(\widehat{FG} \cong \widehat{HJ}\), then \(\angle 1 \cong \angle 2\).
**EXAMPLE 1**

**SPORTS** Refer to the circle graph. Find each measure.

a. \( m\widehat{CD} \)

\( \widehat{CD} \) is a minor arc. \( m\widehat{CD} = m\angle CSD \)

\( \angle CSD \) represents 18% of the whole, or 18% of the circle.

\[ m\angle CSD = 0.18(360) \quad \text{Find 18\% of 360.} \]

\[ = 64.8 \quad \text{Simplify.} \]

b. \( m\widehat{BC} \)

The percents for volleyball and track and field are equal, so the central angles are congruent and the corresponding arcs are congruent.

\[ m\widehat{BC} = m\widehat{CD} = 64.8 \]

**Check Your Progress**

3A. \( m\widehat{EF} \)  
3B. \( m\widehat{FA} \)

**Adjacent arcs** are arcs in a circle that have exactly one point in common. In \( \odot M \), \( \widehat{HJ} \) and \( \widehat{JK} \) are adjacent arcs. As with adjacent angles, you can add the measures of adjacent arcs.

**Postulate 10.1 Arc Addition Postulate**

**Words** The measure of an arc formed by two adjacent arcs is the sum of the measures of the two arcs.

**Example** \( m\widehat{XYZ} = m\widehat{XY} + m\widehat{YZ} \)

**EXAMPLE 4** Use Arc Addition to Find Measures of Arcs

Find each measure in \( \odot F \).

a. \( m\widehat{AED} \)

\[ m\widehat{AED} = m\widehat{AE} + m\widehat{ED} \quad \text{Arc Addition Postulate} \]

\[ = m\angle AFE + m\angle EFD \quad \text{Substitution} \]

\[ = 63 + 90 \text{ or } 153 \]

b. \( m\widehat{ADB} \)

\[ m\widehat{ADB} = m\widehat{AE} + m\widehat{EDB} \quad \text{Arc Addition Postulate} \]

\[ = 63 + 180 \text{ or } 243 \quad \text{\( \widehat{EDB} \) is a semicircle, so \( m\widehat{EDB} = 180 \).} \]

**Check Your Progress**

4A. \( m\widehat{CE} \)  
4B. \( m\widehat{ABD} \)
**Watch Out!**

**Arc Length** The length of an arc is given in linear units, such as centimeters. The measure of an arc is given in degrees.

---

**For Your Foldable**

**Key Concept** Arc Length

**Words** The ratio of the length of an arc $\ell$ to the circumference of the circle is equal to the ratio of the degree measure of the arc to 360.

**Proportion** $\frac{\ell}{2\pi r} = \frac{x}{360}$ or $\ell = \frac{x}{360} \cdot 2\pi r$

---

**EXAMPLE 5** Find Arc Length

Find the length of $\widehat{ZY}$. Round to the nearest hundredth.

**a.**

![Diagram of circle with arc $\widehat{ZY}$ and angle $75^\circ$]

$\ell = \frac{x}{360} \cdot 2\pi r$

$\ell = \frac{75}{360} \cdot 2\pi(4)$

$\approx 5.24$ in. Use a calculator.

**b.**

![Diagram of circle with arc $\widehat{ZY}$ and angle $130^\circ$]

$\ell = \frac{x}{360} \cdot 2\pi r$

$\ell = \frac{130}{360} \cdot 2\pi(5)$

$\approx 11.34$ cm Use a calculator.

**c.**

![Diagram of circle with arc $\widehat{ZY}$ and angle $75^\circ$]

$\ell = \frac{x}{360} \cdot 2\pi r$

$\ell = \frac{75}{360} \cdot 2\pi(6)$

$\approx 7.85$ in. Use a calculator.

Notice that $\widehat{ZY}$ has the same measure, 75, in both Examples 5a and 5c. The arc lengths, however, are different. This is because they are in circles that have different radii.

---

**Check Your Progress**

Find the length of $\widehat{AB}$. Round to the nearest hundredth.

**5A.**

![Diagram of circle with arc $\widehat{AB}$ and angle $45^\circ$]

**5B.**

![Diagram of circle with arc $\widehat{AB}$ and angle $80^\circ$]

**5C.**

![Diagram of circle with arc $\widehat{AB}$ and angle $120^\circ$]

---

**Alternate Method**

The arc lengths in Examples 5a, 5b, and 5c could also have been calculated using the arc length proportion $\frac{\ell}{2\pi r} = \frac{x}{360}$.
Example 1
Find the value of $x$.

1. $60° - 130° = x°$
2. $x° = 140° - 35°$

Example 2
$HK$ and $IG$ are diameters of $\odot L$. Identify each arc as a major arc, minor arc, or semicircle. Then find its measure.

3. $m\widehat{HF}$
4. $m\widehat{HI}$
5. $m\widehat{HGK}$

Example 3
**Example 3**
The graph shows the results of a survey taken by diners relating what is most important about the restaurants where they eat.

- a. Find $m\widehat{AB}$.
- b. Find $m\widehat{BC}$.
- c. Describe the type of arc that the category Great Food represents.

**What Diners Want**

<table>
<thead>
<tr>
<th>Category</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quick Service</td>
<td>8%</td>
</tr>
<tr>
<td>Reasonable Prices</td>
<td>22%</td>
</tr>
<tr>
<td>Great Food</td>
<td>56%</td>
</tr>
<tr>
<td>Atmosphere</td>
<td>11%</td>
</tr>
<tr>
<td>Don't Know</td>
<td>3%</td>
</tr>
</tbody>
</table>

Source: USA TODAY

Example 4
$QS$ is a diameter of $\odot V$. Find each measure.

7. $m\widehat{STP}$
8. $m\widehat{QKT}$
9. $m\widehat{PQR}$

Example 5
Find the length of $\overline{JK}$. Round to the nearest hundredth.

10. $\overline{JK}$
11. $15 \text{ cm}$

Practice and Problem Solving

Example 1
Find the value of $x$.

12. $125° - 155° = x°$
13. $70° - x°$
14. $150° - 85° = x°$
15. $135° - 145° = x°$
Lesson 10-2  Measuring Angles and Arcs

AD and \(CG\) are diameters of \(\odot B\). Identify each arc as a major arc, minor arc, or semicircle. Then find its measure.

16. \(m\overarc{CD}\)  
17. \(m\overarc{AC}\)  
18. \(m\overarc{CG}\)  
19. \(m\overarc{CGD}\)  
20. \(m\overarc{GCF}\)  
21. \(m\overarc{ACD}\)  
22. \(m\overarc{AG}\)  
23. \(m\overarc{ACF}\)

Example 2  p. 693

Example 3  p. 694

24. SHOPPING  The graph shows the results of a survey in which teens were asked where the best place was to shop for clothes.

a. What would be the arc measures associated with the mall and vintage stores categories?

b. Describe the kinds of arcs associated with the first category and the last category.

c. Are there any congruent arcs in this graph? Explain.

25. FOOD  The table shows the results of a survey in which Americans were asked how long food could be on the floor and still be safe to eat.

a. If you were to construct a circle graph of this information, what would be the arc measures associated with the first two categories?

b. Describe the kind of arcs associated with the first category and the last category.

c. Are there any congruent arcs in this graph? Explain.

26. \(m\overarc{FG}\)  
27. \(m\overarc{FH}\)  
28. \(m\overarc{KF}\)  
29. \(m\overarc{FH}\)  
30. \(m\overarc{GKF}\)  
31. \(m\overarc{GHK}\)  
32. \(m\overarc{HK}\)  
33. \(m\overarc{KG}\)  
34. \(m\overarc{KFH}\)  
35. \(m\overarc{HGF}\)  

Examples 2 and 4  pp. 693–694

ENTERTAINMENT  Use the Ferris wheel shown to find each measure.

36. \(\overarc{RS}\), if the radius is 2 inches

37. \(\overarc{QT}\), if the diameter is 9 centimeters

38. \(\overarc{QR}\), if \(PS = 4\) millimeters

39. \(\overarc{RS}\), if \(RT = 15\) inches

40. \(\overarc{QRS}\), if \(RT = 11\) feet

41. \(\overarc{RTS}\), if \(PQ = 3\) meters

Example 5  p. 695

Use \(\odot P\) to find the length of each arc. Round to the nearest hundredth.

36. \(\overarc{RS}\), if the radius is 2 inches

37. \(\overarc{QT}\), if the diameter is 9 centimeters

38. \(\overarc{QR}\), if \(PS = 4\) millimeters

39. \(\overarc{RS}\), if \(RT = 15\) inches

40. \(\overarc{QRS}\), if \(RT = 11\) feet

41. \(\overarc{RTS}\), if \(PQ = 3\) meters

Source: American Diabetic Association

* The length of time the food is on the floor.
**HISTORY**  The figure shows the stars in the Betsy Ross flag referenced at the beginning of the lesson.

42. What is the measure of central angle \( A \)? Explain how you determined your answer.

43. If the diameter of the circle were doubled, what would be the effect on the arc length from the center of one star \( B \) to the next star \( C \)?

44. **FARMS**  The *Pizza Farm* in Madera, California, is a circle divided into eight equal slices, as shown at the right. Each “slice” is used for growing or grazing pizza ingredients.

   a. What is the total arc measure of the slices containing olives, tomatoes, and peppers?

   b. The circle is 125 feet in diameter. What is the arc length of one slice? Round to the nearest hundredth.

Find each measure. Round each linear measure to the nearest hundredth and each arc measure to the nearest degree.

45. circumference of \( \odot S \)

46. \( m\overline{CD} \)

47. radius of \( \odot K \)

48. \( m\widehat{EF} \)

49. \( m\widehat{HD} \)

50. \( m\widehat{HG} \)

**ALGEBRA**  In \( \odot C \), \( m\angle HCG = 2x \) and \( m\angle HCD = 6x + 28 \). Find each measure.

51. **RIDES**  A pirate ship ride follows a semi-circular path, as shown in the diagram.

   a. What is \( m\widehat{AB} \)?

   b. If \( CD = 62 \) feet, what is the length of \( \overline{AB} \)? Round to the nearest hundredth.

52. **PROOF**  Write a two-column proof of Theorem 10.1.

   **Given:** \( \angle BAC \cong \angle DAE \)

   **Prove:** \( \overline{BC} \cong \overline{DE} \)
**53. COORDINATE GEOMETRY** In the graph, point M is located at the origin. Find each measure in \(\odot M\). Round each linear measure to the nearest hundredth and each arc measure to the nearest tenth degree.

a. \(m\overline{L}\)

b. \(m\overparen{KL}\)

c. \(m\overparen{K}\)

d. length of \(\overline{L}\)

e. length of \(\overline{K}\)

**54. MULTIPLE REPRESENTATIONS** In this problem, you will investigate the relationship between chords and arcs.

a. **GEOMETRIC** Draw a circle with congruent chords \(\overline{AB}\) and \(\overline{CD}\). Find the center of the circle. Repeat the process for two additional circles using different chord lengths.

b. **CONCRETE** Cut a piece of patty paper larger than each circle and attach the patty paper at the center of each circle with a push pin. Trace the arc bounded by one of the segments on the patty paper. Rotate the patty paper around the push pin to compare the arc length that you traced to the arc formed by the second segment. Repeat this process for each circle.

c. **VERBAL** Make a conjecture about the relationship between the arcs bounded by congruent chords of a circle. Prove your conjecture.

**H.O.T. Problems** Use Higher-Order Thinking Skills

55. **FIND THE ERROR** Brody says that \(\overparen{WX}\) and \(\overparen{YZ}\) are congruent since their central angles have the same measure. Selena says they are not congruent. Is either of them correct? Explain your reasoning.

**REASONING** Determine whether each statement is sometimes, always, or never true. Explain your reasoning.

56. The measure of a minor arc is less than 180.

57. If a central angle is obtuse, its corresponding arc is a major arc.

58. The sum of the measures of adjacent arcs of a circle depends on the measure of the radius.

59. **CHALLENGE** The measures of \(\overparen{LM}\), \(\overparen{MN}\), and \(\overparen{NL}\) are in the ratio 5:3:4. Find the measure of each arc.

60. **OPEN ENDED** Draw a circle and locate three points on the circle. Estimate the measures of the three nonoverlapping arcs that are formed. Then use a protractor to find the measure of each arc. Label your circle with the arc measures.

61. **CHALLENGE** The time shown on an analog clock is 8:10. What is the measure of the angle formed by the hands of the clock?

62. **WRITING IN MATH** Describe the three different types of arcs in a circle and the method for finding the measure of each one.
63. What is the value of x?
   A 120  
   B 135  
   C 145  
   D 160

64. GRIDDED RESPONSE In ΔB, m∠LBM = 3x and m∠LBQ = 4x + 61. What is the measure of ∠PBQ?

65. ALGEBRA A rectangle's width is represented by x and its length by y. Which expression best represents the area of the rectangle if the length and width are tripled?
   F 3xy  
   G 9xy  
   H (xy)^2  
   J (3xy)^2

66. SAT/ACT What is the area of the shaded region if r = 4?
   A 64 - 16π  
   B 16 - 16π  
   C 16 - 8π  
   D 64 - 8π

67. Name the center of the circle.

68. Identify a chord that is also a diameter.

69. If LN = 12.4, what is JM?

70. Find the image of each polygon with the given vertices after a dilation centered at the origin with the given scale factor. (Lesson 9-6)
   X(-1, 2), Y(2, 1), Z(-1, -2); r = 3
   A(-4, 4), B(4, 4), C(4, -4), D(-4, -4); r = 0.25

71. A(-4, 4), B(4, 4), C(4, -4), D(-4, -4); r = 0.25

72. BASEBALL The diagram shows some dimensions of Comiskey Park in Chicago, Illinois. BD is a segment from home plate to dead center field, and AE is a segment from the left field foul pole to the right field foul pole. If the center fielder is standing at C, how far is he from home plate? (Lesson 8-3)

73. Find x, y, and z. (Lesson 8-1)

74. Find x. (Lesson 8-3)
   24^2 + x^2 = 26^2
   x^2 + 5^2 = 13^2
   30^2 + 35^2 = x^2