1-1 Enrichment

The Tower of Hanoi

The diagram at the right shows the Tower of Hanoi puzzle. Notice that there are three pegs, with a stack of disks on peg a. The object is to move all of the disks to another peg. You may move only one disk at a time and a larger disk may never be put on top of a smaller disk.

As you solve the puzzle, record each move in the table shown. The first two moves are recorded.

Solve.

1. Complete the table to solve the Tower of Hanoi puzzle for three disks.

2. Another way to record each move is to use letters. For example, the first two moves in the table can be recorded as 1c, 2b. This shows that disk 1 is moved to peg c, and then disk 2 is moved to peg b. Record your solution using letters.

3. On a separate sheet of paper, solve the puzzle for four disks. Record your solution.

4. Solve the puzzle for five disks. Record your solution.

5. Suppose you start with an odd number of disks and you want to end with the stack on peg c. What should be your first move?

6. Suppose you start with an even number of disks and you want to end with the stack on peg b. What should be your first move?
The Four Digits Problem

One well-known mathematic problem is to write expressions for consecutive numbers beginning with 1. On this page, you will use the digits 1, 2, 3, and 4. Each digit is used only once. You may use addition, subtraction, multiplication (not division), exponents, and parentheses in any way you wish. Also, you can use two digits to make one number, such as 12 or 34.

Express each number as a combination of the digits 1, 2, 3, and 4.

1 = (3 × 1) − (4 − 2)  
18 = ______________________  
35 = 2\(^{(4+1)}\) + 3

2 = ______________________  
19 = 3(2 + 4) + 1  
36 = ______________________

3 = ______________________  
20 = ______________________  
37 = ______________________

4 = ______________________  
21 = ______________________  
38 = ______________________

5 = ______________________  
22 = ______________________  
39 = ______________________

6 = ______________________  
23 = 31 − (4 × 2)  
40 = ______________________

7 = ______________________  
24 = ______________________  
41 = ______________________

8 = ______________________  
25 = ______________________  
42 = ______________________

9 = ______________________  
26 = ______________________  
43 = 42 + 1\(^3\)

10 = ______________________  
27 = ______________________  
44 = ______________________

11 = ______________________  
28 = ______________________  
45 = ______________________

12 = ______________________  
29 = ______________________  
46 = ______________________

13 = ______________________  
30 = ______________________  
47 = ______________________

14 = ______________________  
31 = ______________________  
48 = ______________________

15 = ______________________  
32 = ______________________  
49 = ______________________

16 = ______________________  
33 = ______________________  
50 = ______________________

17 = ______________________  
34 = ______________________

Does a calculator help in solving these types of puzzles? Give reasons for your opinion.
**Solution Sets**

Consider the following open sentence.

*It* is the name of a month between March and July.

You know that a replacement for the variable *It* must be found in order to determine if the sentence is true or false. If *It* is replaced by either April, May, or June, the sentence is true. The set {April, May, June} is called the solution set of the open sentence given above. This set includes all replacements for the variable that make the sentence true.

**Write the solution set for each open sentence.**

1. It is the name of a state beginning with the letter A.

2. It is a primary color.

3. Its capital is Harrisburg.

4. It is a New England state.

5. \( x + 4 = 10 \)

6. It is the name of a month that contains the letter *r*.

7. During the 1990s, she was the wife of a U.S. President.

8. It is an even number between 1 and 13.

9. \( 31 = 72 - k \)

10. It is the square of 2, 3, or 4.

**Write an open sentence for each solution set.**

11. \{A, E, I, O, U\}

12. \{1, 3, 5, 7, 9\}

13. \{June, July, August\}

14. \{Atlantic, Pacific, Indian, Arctic\}
Closure

A binary operation matches two numbers in a set to just one number. Addition is a binary operation on the set of whole numbers. It matches two numbers such as 4 and 5 to a single number, their sum.

If the result of a binary operation is always a member of the original set, the set is said to be closed under the operation. For example, the set of whole numbers is closed under addition because 4 + 5 is a whole number. The set of whole numbers is not closed under subtraction because 4 − 5 is not a whole number.

Tell whether each operation is binary. Write yes or no.

1. the operation \( \downarrow \), where \( a \downarrow b \) means to choose the lesser number from \( a \) and \( b \)
2. the operation \( \odot \), where \( a \odot b \) means to cube the sum of \( a \) and \( b \)
3. the operation \( \text{sq} \), where \( \text{sq}(a) \) means to square the number \( a \)
4. the operation \( \exp \), where \( \exp(a, b) \) means to find the value of \( a^b \)
5. the operation \( \Uparrow \), where \( a \Uparrow b \) means to match \( a \) and \( b \) to any number greater than either number
6. the operation \( \Rightarrow \), where \( a \Rightarrow b \) means to round the product of \( a \) and \( b \) up to the nearest 10

Tell whether each set is closed under addition. Write yes or no. If your answer is no, give an example.

7. even numbers
8. odd numbers
9. multiples of 3
10. multiples of 5
11. prime numbers
12. nonprime numbers

Tell whether the set of whole numbers is closed under each operation. Write yes or no. If your answer is no, give an example.

13. multiplication: \( a \times b \)
14. division: \( a \div b \)
15. exponentiation: \( ab \)
16. squaring the sum: \( (a + b)^2 \)
Tangram Puzzles

The seven geometric figures shown below are called tans. They are used in a very old Chinese puzzle called tangrams.

Glue the seven tans on heavy paper and cut them out. Use all seven pieces to make each shape shown. Record your solutions below.

1. 2. 

3. 4. 

5. 

6. Each of the two figures shown at the right is made from all seven tans. They seem to be exactly alike, but one has a triangle at the bottom and the other does not. Where does the second figure get this triangle?
Properties of Operations

Let’s make up a new operation and denote it by \( \odot \), so that \( a \odot b \) means \( b^a \).

\[
2 \odot 3 = 3^2 = 9 \\
(1 \odot 2) \odot 3 = 2^1 \odot 3 = 3^2 = 9
\]

1. What number is represented by \( 2 \odot 3 \)?

2. What number is represented by \( 3 \odot 2 \)?

3. Does the operation \( \odot \) appear to be commutative?

4. What number is represented by \( (2 \odot 1) \odot 3 \)?

5. What number is represented by \( 2 \odot (1 \odot 3) \)?

6. Does the operation \( \odot \) appear to be associative?

Let’s make up another operation and denote it by \( \oplus \), so that \( a \oplus b = (a + 1)(b + 1) \).

\[
3 \oplus 2 = (3 + 1)(2 + 1) = 4 \cdot 3 = 12 \\
(1 \oplus 2) \oplus 3 = (2 \cdot 3) \oplus 3 = 6 \oplus 3 = 7 \cdot 4 = 28
\]

7. What number is represented by \( 2 \oplus 3 \)?

8. What number is represented by \( 3 \oplus 2 \)?

9. Does the operation \( \oplus \) appear to be commutative?

10. What number is represented by \( (2 \oplus 3) \oplus 4 \)?

11. What number is represented by \( 2 \oplus (3 \oplus 4) \)?

12. Does the operation \( \oplus \) appear to be associative?

13. What number is represented by \( 1 \oplus (3 \oplus 2) \)?

14. What number is represented by \( (1 \oplus 3) \oplus (1 \oplus 2) \)?

15. Does the operation \( \odot \) appear to be distributive over the operation \( \oplus \)?

16. Let’s explore these operations a little further. What number is represented by \( 3 \oplus (4 \oplus 2) \)?

17. What number is represented by \( (3 \oplus 4) \oplus (3 \oplus 2) \)?

18. Is the operation \( \odot \) actually distributive over the operation \( \oplus \)?
Counterexamples

Some statements in mathematics can be proven false by counterexamples. Consider the following statement.

For any numbers \( a \) and \( b \), \( a - b = b - a \).

You can prove that this statement is false in general if you can find one example for which the statement is false.

Let \( a = 7 \) and \( b = 3 \). Substitute these values in the equation above.

\[
7 - 3 \neq 3 - 7 \\
4 \neq -4
\]

In general, for any numbers \( a \) and \( b \), the statement \( a - b = b - a \) is false. You can make the equivalent verbal statement: subtraction is not a commutative operation.

In each of the following exercises \( a \), \( b \), and \( c \) are any numbers. Prove that the statement is false by counterexample.

1. \( a - (b - c) \neq (a - b) - c \) 
   2. \( a \div (b + c) \neq (a \div b) \div c \) 

3. \( a \div b \neq b \div a \) 
   4. \( a \div (b + c) \neq (a \div b) + (a \div c) \) 

5. \( a + (bc) \neq (a + b)(a + c) \) 
   6. \( a^2 + a^2 \neq a^4 \) 

7. Write the verbal equivalents for Exercises 1, 2, and 3.

8. For the distributive property \( a(b + c) = ab + ac \) it is said that multiplication distributes over addition. Exercises 4 and 5 prove that some operations do not distribute. Write a statement for each exercise that indicates this.
The Digits of π

The number π (pi) is the ratio of the circumference of a circle to its diameter. It is a nonrepeating and nonterminating decimal. The digits of π never form a pattern. Listed at the right are the first 200 digits that follow the decimal point of π.

3.14159 26535 89793 23846
69399 37510 58209 74944
86280 34825 34211 70679
09384 46095 50582 23172
84102 70193 85211 05559
26433 83279 50288 41971
59230 78164 06286 20899
82148 08651 32823 06647
53594 08128 34111 74502
64462 29489 54930 38196

Solve each problem.

1. Suppose each of the digits in π appeared with equal frequency. How many times would each digit appear in the first 200 places following the decimal point?

2. Complete this frequency table for the first 200 digits of π that follow the decimal point.

<table>
<thead>
<tr>
<th>Digit</th>
<th>Frequency (Tally Marks)</th>
<th>Frequency (Number)</th>
<th>Cumulative Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. Explain how the cumulative frequency column can be used to check a project like this one.

4. Which digit(s) appears most often?

5. Which digit(s) appears least often?
Percentiles

The table at the right shows test scores and their frequencies. The frequency is the number of people who had a particular score. The cumulative frequency is the total frequency up to that point, starting at the lowest score and adding up.

<table>
<thead>
<tr>
<th>Score</th>
<th>Frequency</th>
<th>Cumulative Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>95</td>
<td>1</td>
<td>50</td>
</tr>
<tr>
<td>90</td>
<td>2</td>
<td>49</td>
</tr>
<tr>
<td>85</td>
<td>5</td>
<td>47</td>
</tr>
<tr>
<td>80</td>
<td>6</td>
<td>42</td>
</tr>
<tr>
<td>75</td>
<td>7</td>
<td>36</td>
</tr>
<tr>
<td>70</td>
<td>8</td>
<td>29</td>
</tr>
<tr>
<td>65</td>
<td>7</td>
<td>21</td>
</tr>
<tr>
<td>60</td>
<td>6</td>
<td>14</td>
</tr>
<tr>
<td>55</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>50</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>45</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Example 1**

What score is at the 16th percentile?

A score at the 16th percentile means the score just above the lowest 16% of the scores.

16% of the 50 scores is 8 scores.

The 8th score is 55.

The score just above this is 56.

So, the score at the 16th percentile is 56.

Notice that no one had a score of 56 points.

**Example 2**

At what percentile is a score of 75?

There are 29 scores below 75.

Seven scores are at 75. The fourth of these seven is the midpoint of this group.

Adding 4 scores to the 29 gives 33 scores.

33 out of 50 is 66%.

Thus, a score of 75 is at the 66th percentile.

**Use the table above to find the score at each percentile.**

1. 42nd percentile
2. 70th percentile
3. 33rd percentile
4. 90th percentile
5. 58th percentile
6. 80th percentile

**Use the table above to find the percentile of each score.**

7. a score of 50
8. a score of 77
9. a score of 85
10. a score of 58
11. a score of 62
12. a score of 81
Intersection and Union

The intersection of two sets is the set of elements that are in both sets. The intersection of sets A and B is written \( A \cap B \). The union of two sets is the set of elements in either A, B, or both. The union is written \( A \cup B \).

In the drawings below, suppose A is the set of points inside the circle and B is the set of points inside the square. Then, the shaded areas show the intersection in the first drawing and the union in the second drawing.

Write \( A \cap B \) and \( A \cup B \) for each of the following.

1. \( A = \{p, q, r, s, t\} \)
   \( B = \{q, r, s\} \)

2. \( A = \{\text{the integers between 2 and 7}\} \)
   \( B = \{0, 3, 8\} \)

3. \( A = \{\text{the states whose names start with K}\} \)
   \( B = \{\text{the states whose capitals are Honolulu or Topeka}\} \)

4. \( A = \{\text{the positive integer factors of 24}\} \)
   \( B = \{\text{the counting numbers less than 10}\} \)

Suppose \( A = \{\text{numbers } x \text{ such that } x < 3\} \), \( B = \{\text{numbers } x \text{ such as } x \geq -1\} \), and \( C = \{\text{numbers } x \text{ such that } x \leq 1.5\} \). Graph each of the following.

5. \( A \cap B \)

6. \( A \cup B \)

7. \( B \cup C \)

8. \( B \cap C \)

9. \( (A \cap C) \cap B \)

10. \( A \cap (B \cup C) \)
**Rounding Fractions**

Rounding fractions is more difficult than rounding whole numbers or decimals. For example, think about how you would round \( \frac{4}{9} \) inches to the nearest quarter-inch. Through estimation, you might realize that \( \frac{4}{9} \) is less than \( \frac{1}{2} \). But, is it closer to \( \frac{1}{2} \) or to \( \frac{1}{4} \)?

Here are two ways to round fractions. Example 1 uses only the fractions; Example 2 uses decimals.

**Example 1**

Subtract the fraction twice. Use the two nearest quarters.

\[
\frac{1}{2} - \frac{4}{9} = \frac{1}{18} \quad \frac{4}{9} - \frac{1}{4} = \frac{7}{36}
\]

Compare the differences.

\[
\frac{1}{18} < \frac{7}{36}
\]

The smaller difference shows you which fraction to round to.

\( \frac{4}{9} \) rounds to \( \frac{1}{2} \).

**Example 2**

Change the fraction and the two nearest quarters to decimals.

\[
\frac{4}{9} = 0.44, \quad \frac{1}{2} = 0.5, \quad \frac{1}{4} = 0.25
\]

Find the decimal halfway between the two nearest quarters.

\[
\frac{1}{2} (0.5 + 0.25) = 0.375
\]

If the fraction is greater than the halfway decimal, round up. If not, round down.

\[
0.44 > 0.3675. \text{ So, } \frac{4}{9} \text{ is more than half way between } \frac{1}{4} \text{ and } \frac{1}{2}.
\]

\( \frac{4}{9} \) rounds to \( \frac{1}{2} \).

Round each fraction to the nearest one-quarter. Use either method.

1. \( \frac{1}{3} \)
2. \( \frac{3}{7} \)
3. \( \frac{7}{11} \)
4. \( \frac{4}{15} \)
5. \( \frac{7}{20} \)
6. \( \frac{31}{50} \)
7. \( \frac{9}{25} \)
8. \( \frac{23}{30} \)

Round each decimal or fraction to the nearest one-eighth.

9. 0.6
10. 0.1
11. 0.45
12. 0.85
13. \( \frac{5}{7} \)
14. \( \frac{3}{20} \)
15. \( \frac{23}{25} \)
16. \( \frac{5}{9} \)
Compound Interest

In most banks, interest on savings accounts is compounded at set time periods such as three or six months. At the end of each period, the bank adds the interest earned to the account. During the next period, the bank pays interest on all the money in the bank, including interest. In this way, the account earns interest on interest.

Suppose Ms. Tanner has $1000 in an account that is compounded quarterly at 5%. Find the balance after the first two quarters.

Use $I = \frac{p \times r \times t\_{\text{quarter}}}{4}$ to find the interest earned in the first quarter if $p = 1000$ and $r = 5\%$. Why is $t$ equal to $\frac{1}{4}$?

First quarter:  
$I = 1000 \times 0.05 \times \frac{1}{4}$  
$I = 12.50$

The interest, $12.50, earned in the first quarter is added to $1000. The principal becomes $1012.50.

Second quarter:  
$I = 1012.50 \times 0.05 \times \frac{1}{4}$  
$I = 12.65625$

The interest in the second quarter is $12.66.

The balance after two quarters is $1012.50 + 12.66 or $1025.16.

Answer each of the following questions.

1. How much interest is earned in the third quarter of Ms. Tanner’s account?

2. What is the balance in her account after three quarters?

3. How much interest is earned at the end of one year?

4. What is the balance in her account after one year?

5. Suppose Ms. Tanner’s account is compounded semiannually. What is the balance at the end of six months?

6. What is the balance after one year if her account is compounded semiannually?
Other Kinds of Means

There are many different types of means besides the arithmetic mean. A mean for a set of numbers has these two properties:

a. It typifies or represents the set.

b. It is not less than the least number and it is not greater than the greatest number.

Here are the formulas for the arithmetic mean and three other means.

**Arithmetic Mean**
Add the numbers in the set. Then divide the sum by \( n \), the number of elements in the set.

\[
\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}
\]

**Geometric Mean**
Multiply all the numbers in the set. Then find the \( n \)th root of their product.

\[
\sqrt[n]{x_1 \cdot x_2 \cdot x_3 \cdots x_n}
\]

**Harmonic Mean**
Divide the number of elements in the set by the sum of the reciprocals of the numbers.

\[
\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n}}
\]

**Quadratic Mean**
Add the squares of the numbers. Divide their sum by the number in the set. Then, take the square root.

\[
\sqrt{\frac{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2}{n}}
\]

Find the four different means to the nearest hundredth for each set of numbers.

1. 10, 100
2. 50, 60
3. 1, 2, 3, 4, 5,
4. 2, 2, 4, 4

5. Use the results from Exercises 1 to 4 to compare the relative sizes of the four types of means.
2-5 Enrichment

Runs Created

In The 1978 Bill James Baseball Abstract, the author introduced the “runs created” formula.

\[ R = \frac{(h + w)t}{(b + w)} \]

where for each player:

- \( h \) = number of hits
- \( w \) = number of walks,
- \( t \) = number of total bases,
- \( b \) = number of at-bats, and
- \( R \) = approximate number of runs a team scores due to this player’s actions

1. As of June 29, 2001, Roberto Alomar of the Cleveland Indians and Seattle Mariners player Ichiro Suzuki were tied with the highest American League batting average at .351. Find the number of runs created by each player using the data below.

<table>
<thead>
<tr>
<th></th>
<th>( h )</th>
<th>( w )</th>
<th>( t )</th>
<th>( b )</th>
<th>Runs Created</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alomar</td>
<td>97</td>
<td>37</td>
<td>145</td>
<td>276</td>
<td></td>
</tr>
<tr>
<td>Suzuki</td>
<td>121</td>
<td>13</td>
<td>159</td>
<td>345</td>
<td></td>
</tr>
</tbody>
</table>

Based on this information, who do you think is the more valuable American League player? Why?

2. Carlos Lee of the Chicago White Sox and New York Yankee Bernie Williams were both batting .314. Find the number of runs created by each player using the data below.

<table>
<thead>
<tr>
<th></th>
<th>( h )</th>
<th>( w )</th>
<th>( t )</th>
<th>( b )</th>
<th>Runs Created</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lee</td>
<td>81</td>
<td>13</td>
<td>141</td>
<td>258</td>
<td></td>
</tr>
<tr>
<td>Williams</td>
<td>74</td>
<td>31</td>
<td>123</td>
<td>236</td>
<td></td>
</tr>
</tbody>
</table>

3. Why would baseball teams want to calculate the number of runs created by each of their players?
2-6 Enrichment

Geometric Probability

If a dart, thrown at random, hits the triangular board shown at the right, what is the probability that it will hit the shaded region? This can be determined by comparing the area of the shaded region to the area of the entire board. This ratio indicates what fraction of the tosses should hit in the shaded region.

\[
\frac{\text{area of shaded region}}{\text{area of triangular board}} = \frac{\frac{1}{2}(4)(6)}{\frac{1}{2}(8)(6)} = \frac{12}{24} = \frac{1}{2}
\]

In general, if \(S\) is a subregion of some region \(R\), then the probability, \(P(S)\), that a point, chosen at random, belongs to subregion \(S\) is given by the following:

\[
P(S) = \frac{\text{area of subregion } S}{\text{area of region } R}
\]

Find the probability that a point, chosen at random, belongs to the shaded subregions of the following figures.

1. 2. 3. 4. 5. 6. 7. 8. 9.
Scale Drawings

The map at the left below shows building lots for sale. The scale ratio is 1:2400. At the right below is the floor plan for a two-bedroom apartment. The length of the living room is 6 m. On the plan the living room is 6 cm long.

Answer each question.

1. On the map, how many feet are represented by an inch?

2. On the map, measure the frontage of Lot 2 on Sylvan Road in inches. What is the actual frontage in feet?

3. What is the scale ratio represented on the floor plan?

4. On the floor plan, measure the width of the living room in centimeters. What is the actual width in meters?

5. About how many square meters of carpeting would be needed to carpet the living room?

6. Make a scale drawing of your classroom using an appropriate scale.

7. Use your scale drawing to determine how many square meters of tile would be needed to install a new floor in your classroom.
**Rep-Tiles**

A rep-tile is a figure that can be subdivided into smaller copies of itself. The large figure is similar to the small ones and the small figures are all congruent.

Show that each figure is a rep-tile by subdividing it into four smaller and similar figures.

1. ![Figure 1](image1)
2. ![Figure 2](image2)
3. ![Figure 3](image3)
4. ![Figure 4](image4)
5. ![Figure 5](image5)
6. ![Figure 6](image6)

Subdivide each rep-tile into nine smaller and similar figures.

7. ![Figure 7](image7)
8. ![Figure 8](image8)
9. ![Figure 9](image9)
10. ![Figure 10](image10)
Counting-Off Puzzles

Solve each puzzle.

1. Twenty-five people are standing in a circle. Starting with person 1, they count off from 1 to 7 and then start over with 1. Each person who says “7” drops out of the circle. Who is the last person left?

2. Forty people stand in a circle. They count off so that every third person drops out. Which two people are the last ones left?

3. Only half of the 30 students in Sharon’s class can go on a field trip. Sharon arranges the boys and girls as shown. They count off from 1 to 9 and every ninth person drops out until only 15 people are left. Who gets to go on the field trip.

A group of people stand in a circle and count off 1, 2, 1, 2, 1 and so on. Every second person drops out. Person number 1 is the last person left.

4. Draw a diagram to show why the number of people in the circle must be even. Then, explain your answer.

5. When the count returns to person number 1 for the first time, how many people have dropped out?

6. Find the number of people in the circle if the number is between 10 and 20. Do the same if the number is between 30 and 40. What can you conclude about the original number of people?
Dissection Puzzles: Make the Square

In a dissection puzzle, you are to cut apart one figure using only straight cuts and then rearrange the pieces to make a new figure. Usually the puzzle-solver must figure out where to make the given number of cuts. However, for these puzzles, the cut lines are shown. You must discover how to rearrange the pieces.

Cut apart each figure. Then rearrange the pieces to form a square.

1.

2.

3.

4. *Hint:* Cut one of the triangles into two pieces to make this square.
Consecutive Integer Problems

Many types of problems and puzzles involve the idea of consecutive integers. Knowing how to represent these integers algebraically can help to solve the problem.

Example

Find four consecutive odd integers whose sum is $-80$.

An odd integer can be written as $2n + 1$, where $n$ is any integer.

If $2n + 1$ is the first odd integer, then add 2 to get the next largest odd integer, and so on.

Now write an equation to solve this problem.

$$(2n + 1) + (2n + 3) + (2n + 5) + (2n + 7) = -80$$

Exercises

Write an equation for each problem. Then solve.

1. Complete the solution to the problem in the example.

2. Find three consecutive even integers whose sum is 132.

3. Find two consecutive integers whose sum is 19.

4. Find two consecutive integers whose sum is 100.

5. The lesser of two consecutive even integers is 10 more than one-half the greater. Find the integers.

6. The greater of two consecutive even integers is 6 less than three times the lesser. Find the integers.

7. Find four consecutive integers such that twice the sum of the two greater integers exceeds three times the first by 91.

8. Find a set of four consecutive positive integers such that the greatest integer in the set is twice the least integer in the set.
Identities

An equation that is true for every value of the variable is called an identity. When you try to solve an identity, you end up with a statement that is always true. Here is an example.

**Example**

Solve $8 - (5 - 6x) = 3(1 + 2x)$.

$$8 - (5 - 6x) = 3(1 + 2x)$$
$$8 - 5 + 6x = 3 + 6x$$
$$3 + 6x = 3 + 6x$$

**Exercises**

State whether each equation is an identity. If it is not, find its solution.

1. $2(2 - 3x) = 3(3 + x) + 4$
2. $5(m + 1) + 6 = 3(4 + m) + (2m - 1)$

3. $(5t + 9) - (3t - 13) = 2(11 + t)$
4. $14 - (6 - 3c) = 4c - c$

5. $3y - 2(y + 19) = 9y - 3(9 - y)$
6. $3(3h - 1) = 4(h + 3)$

7. Use the true equation $3x - 2 = 3x - 2$ to create an identity of your own.

8. Use the false equation $1 = 2$ to create an equation with no solution.

9. Create an equation whose solution is $x = 3$. 
Angles of a Triangle

In geometry, many statements about physical space are proven to be true. Such statements are called theorems. Here are two examples of geometric theorems.

a. The sum of the measures of the angles of a triangle is 180°.
b. If two sides of a triangle have equal measure, then the two angles opposite those sides also have equal measure.

For each of the triangles, write an equation and then solve for \( x \). (A tick mark on two or more sides of a triangle indicates that the sides have equal measure.)

1. \[
\begin{align*}
\text{Angle 1: } & 70° \\
\text{Angle 2: } & 50° \\
\text{Angle 3: } & x
\end{align*}
\]

2. \[
\begin{align*}
\text{Angle 1: } & x \\
\text{Angle 2: } & 90° \\
\text{Angle 3: } & 45°
\end{align*}
\]

3. \[
\begin{align*}
\text{Angle 1: } & 90° \\
\text{Angle 2: } & x
\end{align*}
\]

4. \[
\begin{align*}
\text{Angle 1: } & x + 30° \\
\text{Angle 2: } & 5x + 10° \\
\text{Angle 3: } & x
\end{align*}
\]

5. \[
\begin{align*}
\text{Angle 1: } & 5x \\
\text{Angle 2: } & 2x \\
\text{Angle 3: } & x
\end{align*}
\]

6. \[
\begin{align*}
\text{Angle 1: } & 4x \\
\text{Angle 2: } & 5x \\
\text{Angle 3: } & 90°
\end{align*}
\]

7. \[
\begin{align*}
\text{Angle 1: } & x - 15° \\
\text{Angle 2: } & x + 30°
\end{align*}
\]

8. \[
\begin{align*}
\text{Angle 1: } & x \\
\text{Angle 2: } & 40°
\end{align*}
\]

9. \[
\begin{align*}
\text{Angle 1: } & x \\
\text{Angle 2: } & x \\
\text{Angle 3: } & x
\end{align*}
\]

10. \[
\begin{align*}
\text{Angle 1: } & 30° \\
\text{Angle 2: } & 2x \\
\text{Angle 3: } & x
\end{align*}
\]

11. Two angles of a triangle have the same measure. The sum of the measures of these angles is one-half the measure of the third angle. Find the measures of the angles of the triangle.

12. The measure of one angle of a triangle is twice the measure of a second angle. The measure of the third angle is 12 less than the sum of the other two. Find the measures of the angles of the triangle.
Using Percent

Use what you have learned about percent to solve each problem.

A TV movie had a “rating” of 15 and a 25 “share.” The rating is the percentage of the nation’s total TV households that were tuned in to this show. The share is the percentage of homes with TVs turned on that were tuned to the movie. How many TV households had their TVs turned off at this time?

To find out, let \( T \) = the number of TV households and \( x \) = the number of TV households with the TV off. Then \( T - x \) = the number of TV households with the TV on.

Since 0.15\( T \) and 0.25\( (T - x) \) both represent the number of households tuned to the movie,

\[
0.15T = 0.25(T - x) \\
0.15T = 0.25T - 0.25x.
\]

Solve for \( x \).
\[
0.25x = 0.10T \\
x = \frac{0.10T}{0.25} = 0.40T
\]

Forty percent of the TV households had their TVs off when the movie was aired.

Answer each question.

1. During that same week, a sports broadcast had a rating of 22.1 and a 43 share. Show that the percent of TV households with their TVs off was about 48.6%.

2. Find the percent of TV households with their TVs turned off during a show with a rating of 18.9 and a 29 share.

3. Show that if \( T \) is the number of TV households, \( r \) is the rating, and \( s \) is the share, then the number of TV households with the TV off is \( \frac{(s - r)T}{s} \).

4. If the fraction of TV households with no TV on is \( \frac{s - r}{s} \) then show that the fraction of TV households with TVs on is \( \frac{r}{s} \).

5. Find the percent of TV households with TVs on during the most watched serial program in history: the last episode of M*A*S*H, which had a 60.3 rating and a 77 share.

6. A local station now has a 2 share. Each share is worth $50,000 in advertising revenue per month. The station is thinking of going commercial free for the three months of summer to gain more listeners. What would its new share have to be for the last 4 months of the year to make more money for the year than it would have made had it not gone commercial free?
Dr. Bernardo Houssay

Even though researchers have been studying the disease diabetes mellitus for hundreds of years, scientists have only recently discovered the cause of the disease and developed methods for reducing its severity. Dr. Bernardo Houssay, an Argentine physiologist, was one of the pioneers of this more modern research. He studied the relationship between diabetes and the pituitary gland, and in 1947 became the first Latin American to win the Nobel Prize in Medicine and Physiology.

Though there is no cure for diabetes, specific diets and exercise can help people control the disease. The American Diabetes Association (ADA) has helped establish flexible dietary guidelines for consumers to follow. These guidelines include some of the following general nutrition rules.

- Fat intake should be equal to or less than 30% of daily calories.
- Saturated fat intake should be equal to or less than 10% of daily calories.
- Protein should be limited to 10% to 20% of daily calories. Persons showing the initial signs of diabetes-induced kidney disease should limit protein to 10% of daily calories.
- Cholesterol intake should be 300 milligrams or less daily.

Refer to the information above for Exercises 1–4.

1. Robert consumed 2100 calories on Tuesday. His fat intake totaled 70 grams, and of that 70 grams, 14 were saturated.
   a. What percentage of his calorie consumption was fat, and what percentage of that fat was saturated? (To find the percentage of calories from fat, multiply the number of fat grams by 9 before dividing by the number of calories.)

   b. Did Robert stay within the recommended allowance of fats?

2. Anna's cholesterol intake was 330 milligrams on Sunday. By what percentage does she need to reduce her cholesterol consumption to remain within the guidelines?

3. What number of fat grams is 30% of 240 calories?

4. Sharon follows a diet that provides about 50 grams of protein each day. Sharon's doctor has just told her to reduce her daily protein intake by 30%. About how much protein will be in her reduced protein diet?
3-9 Enrichment

Diophantine Equations

The first great algebraist, Diophantus of Alexandria (about A.D. 300), devoted much of his work to the solving of indeterminate equations. An indeterminate equation has more than one variable and an unlimited number of solutions. An example is \( x + 2y = 4 \).

When the coefficients of an indeterminate equation are integers and you are asked to find solutions that must be integers, the equation is called *diophantine*. Such equations can be quite difficult to solve, often involving trial and error—and some luck!

Solve each diophantine equation by finding at least one pair of positive integers that makes the equation true. Some hints are given to help you.

1. \( 2x + 5y = 32 \)
   a. First solve the equation for \( x \).
   b. Why must \( y \) be an even number?
   c. Find at least one solution.

2. \( 5x + 2y = 42 \)
   a. First solve the equation for \( x \).
   b. Rewrite your answer in the form \( x = 8 + \) some expression.
   c. Why must \( 2 - 2y \) be a multiple of 5?
   d. Find at least one solution.

3. \( 2x + 7y = 29 \)

4. \( 7x + 5y = 118 \)

5. \( 8x - 13y = 100 \)

6. \( 3x + 4y = 22 \)

7. \( 5x - 14y = 11 \)

8. \( 7x + 3y = 40 \)
Midpoint

The midpoint of a line segment is the point that lies exactly halfway between the two endpoints of the segment. The coordinates of the midpoint of a line segment whose endpoints are \((x_1, y_1)\) and \((x_2, y_2)\) are given by \(\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\).

Find the midpoint of each line segment with the given endpoints.

1. \((7, 1)\) and \((-3, 1)\)
2. \((5, -2)\) and \((9, -8)\)
3. \((-4, 4)\) and \((4, -4)\)
4. \((-3, -6)\) and \((-10, -15)\)

Plot each segment in the coordinate plane. Then find the coordinates of the midpoint.

5. \(JK\) with \(J(5, 2)\) and \(K(-2, -4)\)
6. \(PQ\) with \(P(-1, 4)\) and \(Q(3, -1)\)

You are given the coordinates of one endpoint of a line segment and the midpoint \(M\). Find the coordinates of the other endpoint.

7. \(A(-10, 3)\) and \(M(-6, 7)\)
8. \(D(-1, 4)\) and \(M(3, -6)\)
The Legendary City of Ur

The city of Ur was founded more than five thousand years ago in Mesopotamia (modern-day Iraq). It was one of the world’s first cities. Between 1922 and 1934, archeologists discovered many treasures from this ancient city. A large cemetery from the 26th century B.C. was found to contain large quantities of gold, silver, bronze, and jewels. The many cultural artifacts that were found, such as musical instruments, weapons, mosaics, and statues, have provided historians with valuable clues about the civilization that existed in early Mesopotamia.

1. Suppose that the ordered pairs below represent the volume (cm$^3$) and mass (grams) of ten artifacts from the city of Ur. Plot each point on the graph.
   - A(10, 150)
   - B(150, 1350)
   - C(200, 1760)
   - D(50, 525)
   - E(100, 1500)
   - F(10, 88)
   - G(200, 2100)
   - H(150, 1675)
   - I(100, 900)
   - J(50, 440)

2. The equation relating mass, density, and volume for silver is $m = 10.5V$. Which of the points in Exercise 1 are solutions for this equation?

3. Suppose that the equation $m = 8.8V$ relates mass, density, and volume for the kind of bronze used in the ancient city of Ur. Which of the points in Exercise 1 are solutions for this equation?

4. Explain why the graph in Exercise 1 shows only quadrant 1.
4-3 Enrichment

Inverse Relations

On each grid below, plot the points in Sets A and B. Then connect the points in Set A with the corresponding points in Set B. Then find the inverses of Set A and Set B, plot the two sets, and connect those points.

Set A Set B
(-4, 0) (0, 1)
(-3, 0) (0, 2)
(-2, 0) (0, 3)
(-1, 0) (0, 4)

Set A Set B
(-3, -3) (-2, 1)
(-2, -2) (-1, 2)
(-1, -1) (0, 3)
(0, 0) (1, 4)

Set A Set B
(-4, 1) (3, 2)
(-3, 2) (3, 2)
(-2, 3) (3, 2)
(-1, 4) (3, 2)

13. What is the graphical relationship between the line segments you drew connecting points in Sets A and B and the line segments connecting points in the inverses of those two sets?
Coordinate Geometry and Area

How would you find the area of a triangle whose vertices have the coordinates $A(-1, 2)$, $B(1, 4)$, and $C(3, 0)$?

When a figure has no sides parallel to either axis, the height and base are difficult to find.

One method of finding the area is to enclose the figure in a rectangle and subtract the area of the surrounding triangles from the area of the rectangle.

Area of rectangle $DEFC$  
$= 4 \times 4$  
$= 16$ square units

Area of triangle I $= \frac{1}{2} (2)(4) = 4$

Area of triangle II $= \frac{1}{2} (2)(4) = 4$

Area of triangle III $= \frac{1}{2} (2)(2) = 2$

Total $= 10$ square units

Area of triangle $ABC = 16 - 10$, or $6$ square units

Find the areas of the figures with the following vertices.

1. $A(-4, -6), B(0, 4), \quad C(4, 2)$

2. $A(6, -2), B(8, -10), \quad C(12, -6)$

3. $A(0, 2), B(2, 7), \quad C(6, 10), D(9, -2)$
**Taxicab Graphs**

You have used a rectangular coordinate system to graph equations such as $y = x - 1$ on a coordinate plane. In a coordinate plane, the numbers in an ordered pair $(x, y)$ can be any two real numbers.

A **taxicab plane** is different from the usual coordinate plane. The only points allowed are those that exist along the horizontal and vertical grid lines. You may think of the points as taxicabs that must stay on the streets.

The taxicab graph shows the equations $y = -2$ and $y = x - 1$. Notice that one of the graphs is no longer a straight line. It is now a collection of separate points.

**Graph these equations on the taxicab plane at the right.**

1. $y = x + 1$
2. $y = -2x + 3$
3. $y = 2.5$
4. $x = -4$

**Use your graphs for these problems.**

5. Which of the equations has the same graph in both the usual coordinate plane and the taxicab plane?

6. Describe the form of equations that have the same graph in both the usual coordinate plane and the taxicab plane.

**In the taxicab plane, distances are not measured diagonally, but along the streets. Write the taxi-distance between each pair of points.**

7. $(0, 0)$ and $(5, 2)$
8. $(0, 0)$ and $(3, 2)$
9. $(0, 0)$ and $(2, 1.5)$
10. $(1, 2)$ and $(4, 3)$
11. $(2, 4)$ and $(-1, 3)$
12. $(0, 4)$ and $(-2, 0)$

**Draw these graphs on the taxicab grid at the right.**

13. The set of points whose taxi-distance from $(0, 0)$ is 2 units.
14. The set of points whose taxi-distance from $(2, 1)$ is 3 units.
Composite Functions

Three things are needed to have a function—a set called the domain, a set called the range, and a rule that matches each element in the domain with only one element in the range. Here is an example.

Rule: \( f(x) = 2x + 1 \)

\[
\begin{align*}
  f(1) &= 2(1) + 1 = 3 \\
  f(2) &= 2(2) + 1 = 5 \\
  f(-3) &= 2(-3) + 1 = -5
\end{align*}
\]

Suppose we have three sets A, B, and C and two functions described as shown below.

Rule: \( f(x) = 2x + 1 \) Rule: \( g(y) = 3y - 4 \)

\[
\begin{align*}
  g(y) &= 3y - 4 \\
  g(3) &= 3(3) - 4 = 5
\end{align*}
\]

Let's find a rule that will match elements of set A with elements of set C without finding any elements in set B. In other words, let's find a rule for the composite function \( g[f(x)] \).

Since \( f(x) = 2x + 1 \), \( g[f(x)] = g(2x + 1) \).

Since \( g(y) = 3y - 4 \), \( g(2x + 1) = 3(2x + 1) - 4 = 6x - 1 \).

Therefore, \( g[f(x)] = 6x - 1 \).

Find a rule for the composite function \( g[f(x)] \).

1. \( f(x) = 3x \) and \( g(y) = 2y + 1 \)
2. \( f(x) = x^2 + 1 \) and \( g(y) = 4y \)
3. \( f(x) = -2x \) and \( g(y) = y^2 - 3y \)
4. \( f(x) = \frac{1}{x - 3} \) and \( g(y) = y^{-1} \)
5. Is it always the case that \( g[f(x)] = f[g(x)] \)? Justify your answer.
Arithmetic Series

An arithmetic series is a series in which each term after the first may be found by adding the same number to the preceding term. Let $S$ stand for the following series in which each term is 3 more than the preceding one.

$$S = 2 + 5 + 8 + 11 + 14 + 17 + 20$$

The series remains the same if we reverse the order of all the terms. So let us reverse the order of the terms and add one series to the other, term by term. This is shown at the right.

Let $a$ represent the first term of the series.
Let $\ell$ represent the last term of the series.
Let $n$ represent the number of terms in the series.

In the preceding example, $a = 2, \ell = 20$, and $n = 7$. Notice that when you add the two series, term by term, the sum of each pair of terms is 22. That sum can be found by adding the first and last terms, $2 + 20$ or $a + \ell$. Notice also that there are 7, or $n$, such sums. Therefore, the value of $2S$ is 7(22), or $n(a + \ell)$ in the general case. Since this is twice the sum of the series, you can use the following formula to find the sum of any arithmetic series.

$$S = \frac{n(a + \ell)}{2}$$

**Example 1**

Find the sum: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$

$a = 1, \ell = 9, n = 9$, so $S = \frac{9(1 + 9)}{2} = \frac{9 \cdot 10}{2} = 45$

**Example 2**

Find the sum: $-9 + (-5) + (-1) + 3 + 7 + 11 + 15$

$a = 29, \ell = 15, n = 7$, so $S = \frac{7(-9 + 15)}{2} = \frac{7 \cdot 6}{2} = 21$

**Find the sum of each arithmetic series.**

1. $3 + 6 + 9 + 12 + 15 + 18 + 21 + 24$
2. $10 + 15 + 20 + 25 + 30 + 35 + 40 + 45 + 50$
3. $-21 + (-16) + (-11) + (-6) + (-1) + 4 + 9 + 14$
4. even whole numbers from 2 through 100
5. odd whole numbers between 0 and 100
Traceable Figures

Try to trace over each of the figures below without tracing the same segment twice.

The figure at the left cannot be traced, but the one at the right can. The rule is that a figure is traceable if it has no points, or exactly two points where an odd number of segments meet. The figure at the left has three segments meeting at each of the four corners. However, the figure at the right has exactly two points, L and Q, where an odd number of segments meet.

Determine whether each figure can be traced. If it can, then name the starting point and number the sides in the order in which they should be traced.

1. E
   ABC
   D
   F
   H JG
   K

2. T
   U
   P
   V
   S
   R

3. E
   W X Y

4. A
   B
   C
   D
   E
   F
   G
   H J K

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Treasure Hunt with Slopes

Using the definition of slope, draw lines with the slopes listed below. A correct solution will trace the route to the treasure.

1. 3  
2. $\frac{1}{4}$  
3. $-\frac{2}{5}$  
4. 0  
5. 1  
6. $-1$  
7. no slope  
8. $\frac{2}{7}$  
9. $\frac{3}{2}$  
10. $\frac{1}{3}$  
11. $-\frac{3}{4}$  
12. 3
nth Power Variation

An equation of the form $y = kx^n$, where $k \neq 0$, describes an $n$th power variation. The variable $n$ can be replaced by 2 to indicate the second power of $x$ (the square of $x$) or by 3 to indicate the third power of $x$ (the cube of $x$).

Assume that the weight of a person of average build varies directly as the cube of that person’s height. The equation of variation has the form $w = kh^3$.

The weight that a person’s legs will support is proportional to the cross-sectional area of the leg bones. This area varies directly as the square of the person’s height. The equation of variation has the form $s = kh^2$.

Answer each question.

1. For a person 6 feet tall who weighs 200 pounds, find a value for $k$ in the equation $w = kh^3$.

2. Use your answer from Exercise 1 to predict the weight of a person who is 5 feet tall.

3. Find the value for $k$ in the equation $w = kh^3$ for a baby who is 20 inches long and weighs 6 pounds.

4. How does your answer to Exercise 3 demonstrate that a baby is significantly fatter in proportion to its height than an adult?

5. For a person 6 feet tall who weighs 200 pounds, find a value for $k$ in the equation $s = kh^2$.

6. For a baby who is 20 inches long and weighs 6 pounds, find an “infant value” for $k$ in the equation $s = kh^2$.

7. According to the adult equation you found (Exercise 1), how much would an imaginary giant 20 feet tall weigh?

8. According to the adult equation for weight supported (Exercise 5), how much weight could a 20-foot tall giant’s legs actually support?

9. What can you conclude from Exercises 7 and 8?
Relating Slope-Intercept Form and Standard Forms

You have learned that slope can be defined in terms of \( \frac{\text{rise}}{\text{run}} \) or \( \frac{y_2 - y_1}{x_2 - x_1} \).

Another definition can be found from the standard form of a linear equation. Standard form is \( Ax + By = C \), where \( A, B, \) and \( C \) are integers, \( A \geq 0 \), and \( A \) and \( B \) are not both zero.

1. Solve \( Ax + By = C \) for \( y \). Your answer should be written in slope-intercept form.

2. Use the slope-intercept equation you wrote in Exercise 1 to write expressions for the slope and the \( y \)-intercept in terms of \( A, B, \) and \( C \).

Use the expressions in Exercise 2 above to find the slope and \( y \)-intercept of each equation.

3. \( 2x + y = -4 \) 

4. \( 4x + 3y = 24 \)

5. \( 4x + 6y = -36 \) 

6. \( x - 3y = -27 \)

7. \( x - 2y = 6 \) 

8. \( 4y = 20 \)
Celsius and Kelvin Temperatures

If you blow up a balloon and put it in the refrigerator, the balloon will shrink as the temperature of the air in the balloon decreases.

The volume of a certain gas is measured at 30° Celsius. The temperature is decreased and the volume is measured again.

<table>
<thead>
<tr>
<th>Temperature (°C)</th>
<th>Volume (mL)</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>202 mL</td>
</tr>
<tr>
<td>21°</td>
<td>196 mL</td>
</tr>
<tr>
<td>0°</td>
<td>182 mL</td>
</tr>
<tr>
<td>−12°</td>
<td>174 mL</td>
</tr>
<tr>
<td>−27°</td>
<td>164 mL</td>
</tr>
</tbody>
</table>

1. Graph this table on the coordinate plane provided below.

2. Find the equation of the line that passes through the points you graphed in Exercise 1.

3. Use the equation you found in Exercise 2 to find the temperature that would give a volume of zero. This temperature is the lowest one possible and is called absolute zero.

4. In 1848, Lord Kelvin proposed a new temperature scale with 0 being assigned to absolute zero. The size of the degree chosen was the same size as the Celsius degree. Change each of the Celsius temperatures in the table above to degrees Kelvin.
Collinearity

You have learned how to find the slope between two points on a line. Does it matter which two points you use? How does your choice of points affect the slope-intercept form of the equation of the line?

1. Choose three different pairs of points from the graph at the right. Write the slope-intercept form of the line using each pair.

2. How are the equations related?

3. What conclusion can you draw from your answers to Exercises 1 and 2?

When points are contained in the same line, they are said to be collinear. Even though points may look like they form a straight line when connected, it does not mean that they actually do. By checking pairs of points on a line you can determine whether the line represents a linear relationship.

4. Choose several pairs of points from the graph at the right and write the slope-intercept form of the line using each pair.

5. What conclusion can you draw from your equations in Exercise 4? Is this a straight line?
**Pencils of Lines**

All of the lines that pass through a single point in the same plane are called a **pencil of lines**.

All lines with the same slope, but different intercepts, are also called a “pencil,” a **pencil of parallel lines**.

**Graph some of the lines in each pencil.**

1. A pencil of lines through the point (1, 3)

2. A pencil of lines described by \( y - 4 = m(x - 2) \), where \( m \) is any real number

3. A pencil of lines parallel to the line \( x - 2y = 7 \)

4. A pencil of lines described by \( y = mx + 3m - 2 \)
Latitude and Temperature

The latitude of a place on Earth is the measure of its distance from the equator. What do you think is the relationship between a city’s latitude and its January temperature? At the right is a table containing the latitudes and January mean temperatures for fifteen U.S. cities.

<table>
<thead>
<tr>
<th>U.S. City</th>
<th>Latitude</th>
<th>January Mean Temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td>Albany, New York</td>
<td>42:40 N</td>
<td>20.7°F</td>
</tr>
<tr>
<td>Albuquerque, New Mexico</td>
<td>35:07 N</td>
<td>34.3°F</td>
</tr>
<tr>
<td>Anchorage, Alaska</td>
<td>61:11 N</td>
<td>14.9°F</td>
</tr>
<tr>
<td>Birmingham, Alabama</td>
<td>33:32 N</td>
<td>41.7°F</td>
</tr>
<tr>
<td>Charleston, South Carolina</td>
<td>32:47 N</td>
<td>47.1°F</td>
</tr>
<tr>
<td>Chicago, Illinois</td>
<td>41:50 N</td>
<td>21.0°F</td>
</tr>
<tr>
<td>Columbus, Ohio</td>
<td>39:59 N</td>
<td>26.3°F</td>
</tr>
<tr>
<td>Duluth, Minnesota</td>
<td>46:47 N</td>
<td>7.0°F</td>
</tr>
<tr>
<td>Fairbanks, Alaska</td>
<td>64:50 N</td>
<td>−10.1°F</td>
</tr>
<tr>
<td>Galveston, Texas</td>
<td>29:14 N</td>
<td>52.9°F</td>
</tr>
<tr>
<td>Honolulu, Hawaii</td>
<td>21:19 N</td>
<td>72.9°F</td>
</tr>
<tr>
<td>Las Vegas, Nevada</td>
<td>36:12 N</td>
<td>45.1°F</td>
</tr>
<tr>
<td>Miami, Florida</td>
<td>25:47 N</td>
<td>67.3°F</td>
</tr>
<tr>
<td>Richmond, Virginia</td>
<td>37:32 N</td>
<td>35.8°F</td>
</tr>
<tr>
<td>Tucson, Arizona</td>
<td>32:12 N</td>
<td>51.3°F</td>
</tr>
</tbody>
</table>

Sources: www.indo.com and www.nws.noaa.gov/climatex.html

1. Use the information in the table to create a scatter plot and draw a line of best fit for the data.

2. Write an equation for the line of fit. Make a conjecture about the relationship between a city’s latitude and its mean January temperature.

3. Use your equation to predict the January mean temperature of Juneau, Alaska, which has latitude 58:23 N.

4. What would you expect to be the latitude of a city with a January mean temperature of 15°F?

5. Was your conjecture about the relationship between latitude and temperature correct?

6. Research the latitudes and temperatures for cities in the southern hemisphere instead. Does your conjecture hold for these cities as well?
**Triangle Inequalities**

Recall that a line segment can be named by the letters of its endpoints. Line segment $AB$ (written as $\overline{AB}$) has points $A$ and $B$ for endpoints. The *length* of $AB$ is written without the bar as $AB$.

$$AB < BC \quad m\angle A < m\angle B$$

The statement on the left above shows that $AB$ is shorter than $BC$.

The statement on the right above shows that the measure of angle $A$ is less than that of angle $B$.

These three inequalities are true for any triangle $ABC$, no matter how long the sides.

- **a.** $AB + BC > AC$
- **b.** If $AB > AC$, then $m\angle C > m\angle B$.
- **c.** If $m\angle C > m\angle B$, then $AB > AC$.

**Use the three triangle inequalities for these problems.**

1. List the sides of triangle $DEF$ in order of increasing length.

2. In the figure at the right, which line segment is the shortest?

3. Explain why the lengths 5 cm, 10 cm, and 20 cm could not be used to make a triangle.

4. Two sides of a triangle measure 3 in. and 7 in. Between which two values must the third side be?

5. In triangle $XYZ$, $XY = 15$, $YZ = 12$, and $XZ = 9$. Which is the greatest angle? Which is the least?

6. List the angles $\angle A$, $\angle C$, $\angle ABC$, and $\angle ABD$, in order of increasing size.
The Maya

The Maya were a Native American people who lived from about 1500 B.C. to about 1500 A.D. in the region that today encompasses much of Central America and southern Mexico. Their many accomplishments include exceptional architecture, pottery, painting, and sculpture, as well as significant advances in the fields of astronomy and mathematics.

The Maya developed a system of numeration that was based on the number twenty. The basic symbols of this system are shown in the table at the right. The places in a Mayan numeral are written vertically—the bottom place represents ones, the place above represents twenties, the place above that represents 20 \times 20, or four hundreds, and so on. For instance, this is how to write the number 997 in Mayan numerals.

\[
\begin{align*}
2 \times 400 & = 800 \\
9 \times 20 & = 180 \\
17 \times 1 & = 17 \\
\text{997}
\end{align*}
\]

Evaluate each expression when \(v = \), \(w = \), \(x = \), \(y = \), and \(z = \). Then write the answer in Mayan numerals. Exercise 5 is done for you.

1. \(\frac{z}{w}\)  

2. \(\frac{v + w + z}{x}\)  

3. \(xv\)  

4. \(vxy\)  

5. \(wx - z\)  

6. \(vz + xy\)  

7. \(w(v + x + z)\)  

8. \(vwz\)  

9. \(z(wx - x)\)

Tell whether each statement is true or false.

10. \(\text{****} + \text{••} = \text{••} + \text{****}\)  

11. \(\text{•} = \text{•}\)  

12. \(\text{••} = \text{••\•}\)  

13. \((\text{••} + \text{•••}) + \text{•••} = \text{•••} + (\text{•••} + \text{•••})\)

14. How are Exercises 10 and 11 alike? How are they different?
Carlos Montezuma

During his lifetime, Carlos Montezuma (1865?–1923) was one of the most influential Native Americans in the United States. He was recognized as a prominent physician and was also a passionate advocate of the rights of Native American peoples. The exercises that follow will help you learn some interesting facts about Dr. Montezuma’s life.

Solve each inequality. The word or phrase next to the equivalent inequality will complete the statement correctly.

1. \[-2k > 10\]
   Montezuma was born in the state of ___.
   a. \(k < -5\) Arizona
   b. \(k > -5\) Montana
   c. \(k > 12\) Utah

2. \[5 \geq r - 9\]
   He was a Native American of the Yavapais, who are a ___ people.
   a. \(r \leq -4\) Navajo
   b. \(r \geq -4\) Mohawk
   c. \(r \leq 14\) Mohave-Apache

3. \[-y \leq -9\]
   Montezuma received a medical degree from ___ in 1889.
   a. \(y \geq 9\) Chicago Medical College
   b. \(y \geq -9\) Harvard Medical School
   c. \(y \leq 9\) Johns Hopkins University

4. \[-3 + q > 12\]
   As a physician, Montezuma’s field of specialization was ___.
   a. \(q > -4\) heart surgery
   b. \(q > 15\) internal medicine
   c. \(q < -15\) respiratory diseases

5. \[5 + 4x - 14 \leq x\]
   For much of his career, he maintained a medical practice in ___.
   a. \(x \leq 9\) New York City
   b. \(x \leq 3\) Chicago
   c. \(x \geq -9\) Boston

6. \[7 - t < 7 + t\]
   In addition to maintaining his medical practice, he was also a(n) ___.
   a. \(t > 7\) director of a blood bank
   b. \(t > 0\) instructor at a medical college
   c. \(t < -7\) legal counsel to physicians

7. \[3a + 8 \geq 4a - 10\]
   Montezuma founded, wrote, and edited ___, a monthly newsletter that addressed Native American concerns.
   a. \(a \leq -2\) Yavapai
   b. \(a \geq 18\) Apache
   c. \(a \leq 18\) Wassaja

8. \[6n > 8n - 12\]
   Montezuma testified before a committee of the United States Congress concerning his work in treating ___.
   a. \(n < 6\) appendicitis
   b. \(n > -6\) asthma
   c. \(n > -10\) heart attacks
Some Properties of Inequalities

The two expressions on either side of an inequality symbol are sometimes called the first and second members of the inequality.

If the inequality symbols of two inequalities point in the same direction, the inequalities have the same sense. For example, \(a < b\) and \(c < d\) have the same sense; \(a < b\) and \(c > d\) have opposite senses.

In the problems on this page, you will explore some properties of inequalities.

Three of the four statements below are true for all numbers \(a\) and \(b\) (or \(a, b, c,\) and \(d\)). Write each statement in algebraic form. If the statement is true for all numbers, prove it. If it is not true, give an example to show that it is false.

1. Given an inequality, a new and equivalent inequality can be created by interchanging the members and reversing the sense.

2. Given an inequality, a new and equivalent inequality can be created by changing the signs of both terms and reversing the sense.

3. Given two inequalities with the same sense, the sum of the corresponding members are members of an equivalent inequality with the same sense.

4. Given two inequalities with the same sense, the difference of the corresponding members are members of an equivalent inequality with the same sense.
**Precision of Measurement**

The precision of a measurement depends both on your accuracy in measuring and the number of divisions on the ruler you use. Suppose you measured a length of wood to the nearest one-eighth of an inch and got a length of $6\frac{5}{8}$ in.

The drawing shows that the actual measurement lies somewhere between $6\frac{9}{11}$ in. and $6\frac{11}{16}$ in. This measurement can be written using the symbol $\pm$, which is read *plus or minus*. It can also be written as a compound inequality.

$$6\frac{5}{8} \pm \frac{1}{16} \text{ in.} \quad 6\frac{9}{16} \text{ in.} \leq m \leq 6\frac{11}{16} \text{ in.}$$

In this example, $\frac{1}{16}$ in. is the absolute error. The absolute error is one-half the smallest unit used in a measurement.

**Write each measurement as a compound inequality. Use the variable m.**

1. $3\frac{1}{2} \pm \frac{1}{4}$ in.  
2. $9.78 \pm 0.005$ cm  
3. $2.4 \pm 0.05$ g  
4. $28 \pm \frac{1}{2}$ ft  
5. $15 \pm 0.5$ cm  
6. $\frac{11}{16} \pm \frac{1}{64}$ in.

**For each measurement, give the smallest unit used and the absolute error.**

7. $12.5 \text{ cm} \leq m \leq 13.5 \text{ cm}$  
8. $12\frac{1}{8} \text{ in.} \leq m \leq 12\frac{3}{8} \text{ in.}$  
9. $56\frac{1}{2} \text{ in.} \leq m \leq 57\frac{1}{2} \text{ in.}$  
10. $23.05 \text{ mm} \leq m \leq 23.15 \text{ mm}$
Using Equations: Ideal Weight

You can find your ideal weight as follows.

A woman should weigh 100 pounds for the first 5 feet of height and 5 additional pounds for each inch over 5 feet (5 feet = 60 inches).
A man should weigh 106 pounds for the first 5 feet of height and 6 additional pounds for each inch over 5 feet. These formulas apply to people with normal bone structures.

To determine your bone structure, wrap your thumb and index finger around the wrist of your other hand. If the thumb and finger just touch, you have normal bone structure. If they overlap, you are small-boned. If they don’t overlap, you are large-boned. Small-boned people should decrease their calculated ideal weight by 10%. Large-boned people should increase the value by 10%.

Calculate the ideal weights of these people.

1. woman, 5 ft 4 in., normal-boned
2. man, 5 ft 11 in., large-boned
3. man, 6 ft 5 in., small-boned
4. you, if you are at least 5 ft tall

For Exercises 5–9, use the following information.

Suppose a normal-boned man is \(x\) inches tall. If he is at least 5 feet tall, then \(x - 60\) represents the number of inches this man is over 5 feet tall. For each of these inches, his ideal weight is increased by 6 pounds. Thus, his proper weight \((y)\) is given by the formula \(y = 6(x - 60) + 106\) or \(y = 6x - 254\). If the man is large-boned, the formula becomes \(y = 6x - 254 + 0.10(6x - 254)\).

5. Write the formula for the weight of a large-boned man in slope-intercept form.

6. Derive the formula for the ideal weight \((y)\) of a normal-boned female with height \(x\) inches. Write the formula in slope-intercept form.

7. Derive the formula in slope-intercept form for the ideal weight \((y)\) of a large-boned female with height \(x\) inches.

8. Derive the formula in slope-intercept form for the ideal weight \((y)\) of a small-boned male with height \(x\) inches.

9. Find the heights at which normal-boned males and large-boned females would weigh the same.
Graphing a Trip

The distance formula, \( d = rt \), is used to solve many types of problems. If you graph an equation such as \( d = 50t \), the graph is a model for a car going at 50 mi/h. The time the car travels is \( t \); the distance in miles the car covers is \( d \). The slope of the line is the speed.

Suppose you drive to a nearby town and return. You average 50 mi/h on the trip out but only 25 mi/h on the trip home. The round trip takes 5 hours. How far away is the town?

The graph at the right represents your trip. Notice that the return trip is shown with a negative slope because you are driving in the opposite direction.

Solve each problem.

1. Estimate the answer to the problem in the above example. About how far away is the town?

2. Graph this trip and solve the problem. An airplane has enough fuel for 3 hours of safe flying. On the trip out the pilot averages 200 mi/h flying against a headwind. On the trip back, the pilot averages 250 mi/h. How long a trip out can the pilot make?

3. Graph this trip and solve the problem. You drive to a town 100 miles away. On the trip out you average 25 mi/h. On the trip back you average 50 mi/h. How many hours do you spend driving?

4. Graph this trip and solve the problem. You drive at an average speed of 50 mi/h to a discount shopping plaza, spend 2 hours shopping, and then return at an average speed of 25 mi/h. The entire trip takes 8 hours. How far away is the shopping plaza?
**Equations of Lines and Planes in Intercept Form**

One form that a linear equation may take is intercept form. The constants $a$ and $b$ are the $x$- and $y$-intercepts of the graph.

$$\frac{x}{a} + \frac{y}{b} = 1$$

In three-dimensional space, the equation of a plane takes a similar form.

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Here, the constants $a$, $b$, and $c$ are the points where the plane meets the $x$, $y$, and $z$-axes.

**Solve each problem.**

1. Graph the equation $\frac{x}{3} + \frac{y}{2} + \frac{z}{1} = 1$.

2. For the plane in Exercise 1, write an equation for the line where the plane intersects the $xy$-plane. Use intercept forms.

3. Write an equation for the line where the plane intersects the $xz$-plane.

4. Write an equation for the line where the plane intersects the $yz$-plane.

5. Graph the equation $\frac{x}{1} + \frac{y}{4} + \frac{z}{2} = 1$.

6. Write an equation for the $xy$-plane.

7. Write an equation for the $yz$-plane.

8. Write an equation for a plane parallel to the $xy$-plane with a $z$-intercept of 2.

9. Write an equation for a plane parallel to the $yz$-plane with an $x$-intercept of 23.
Rózsa Péter

Rózsa Péter (1905–1977) was a Hungarian mathematician dedicated to teaching others about mathematics. As professor of mathematics at a teachers’ college in Budapest, she wrote several mathematics textbooks and championed reforms in the teaching of mathematics. In 1945 she wrote *Playing with Infinity: Mathematical Explorations and Excursions*, a popular work in which she attempted to convey the spirit of mathematics to the general public.

By far Péter’s greatest contribution to mathematics was her pioneering research in the field of recursive function theory. When you evaluate a function *recursively*, you begin with one initial value of \( x \). Working from this single number, you can use the function to generate an entire sequence of numbers. For instance, here is how you use an initial value of \( x = 1 \) to evaluate the function \( f(x) = 3x \) recursively.

\[
\begin{align*}
  f(1) &= 3(1) = 3 \\
  \downarrow \\
  f(3) &= 3(3) = 9 \\
  \downarrow \\
  f(9) &= 3(9) = 27 \\
  \downarrow \\
  f(27) &= 3(27) = 81 \\
  \downarrow \\
  f(81) &= 3(81) = 243
\end{align*}
\]

The first five numbers of the sequence generated by this function are 3, 9, 27, 81, and 243.

**Write the first five numbers of the sequence generated by each function, using the given number as the initial value of \( x \).**

1. \( f(x) = 3x; \ x = 2 \) 
2. \( g(x) = x - 5; \ x = 1 \)

3. \( f(x) = 2x + 1; \ x = -3 \) 
4. \( f(x) = x^2; \ x = 2 \)

5. \( h(x) = -x; \ x = 3 \) 
6. \( k(x) = \frac{1}{x}; \ x = 10 \)
George Washington Carver and Percy Julian

In 1990, George Washington Carver and Percy Julian became the first African Americans elected to the National Inventors Hall of Fame. Carver (1864–1943) was an agricultural scientist known worldwide for developing hundreds of uses for the peanut and the sweet potato. His work revitalized the economy of the southern United States because it was no longer dependent solely upon cotton. Julian (1898–1975) was a research chemist who became famous for inventing a method of making a synthetic cortisone from soybeans. His discovery has had many medical applications, particularly in the treatment of arthritis.

There are dozens of other African American inventors whose accomplishments are not as well known. Their inventions range from common household items like the ironing board to complex devices that have revolutionized manufacturing. The exercises that follow will help you identify just a few of these inventors and their inventions.

Match the inventors with their inventions by matching each system with its solution. (Not all the solutions will be used.)

<table>
<thead>
<tr>
<th>Inventor</th>
<th>Equations</th>
<th>Solution</th>
<th>Invention</th>
</tr>
</thead>
</table>
| 1. Sara Boone    | \[ x + y = 2 \]
                  | \[ x - y = 10 \]                                                         | (1, 4)      | automatic traffic signal                            |
| 2. Sarah Goode   | \[ x = 2 - y \]
                  | \[ 2y + x = 9 \]                                                         | (4, -2)     | eggbeater                                          |
| 3. Frederick M.  | \[ y = 2x + 6 \]
                  | \[ y = -x - 3 \]                                                         | (-2, 3)     | fire extinguisher                                   |
| Jones            |                                                                          |             |                                                     |
| 4. J. L. Love    | \[ 2x + 3y = 8 \]
                  | \[ 2x - y = -8 \]                                                        | (-5, 7)     | folding cabinet bed                                 |
| 5. T. J. Marshall| \[ y - 3x = 9 \]
                  | \[ 2y + x = 4 \]                                                         | (6, -4)     | ironing board                                       |
| 6. Jan Matzeliger| \[ y + 4 = 2x \]
                  | \[ 6x - 3y = 12 \]                                                       | (-2, 4)     | pencil sharpener                                    |
| 7. Garrett A.    | \[ 3x - 2y = -5 \]
                  | \[ 3y - 4x = 8 \]                                                        | (-3, 0)     | portable X-ray machine                              |
| Morgan           |                                                                          |             |                                                     |
| 8. Norbert Rillieux| \[ 3x - y = 12 \]
                        | \[ y - 3x = 15 \]                                                        | (2, -3)     | player piano                                        |
|                  |                                                                          |             |                                                     |
|                  |                                                                          |             | L. no solution evaporating pan for refining sugar  |
|                  |                                                                          |             | J. infinitely many solutions lasting (shaping)      |
|                  |                                                                          |             | machine for manufacturing shoes                     |
Describing Regions

The shaded region inside the triangle can be described with a system of three inequalities.

\[
\begin{align*}
    y &< 2x + 1 \\
    y &> \frac{1}{3}x - 3 \\
    y &> 29x - 31
\end{align*}
\]

Write systems of inequalities to describe each region. You may first need to divide a region into triangles or quadrilaterals.

1. 

2. 

3.
An Wang

An Wang (1920–1990) was an Asian-American who became one of the pioneers of the computer industry in the United States. He grew up in Shanghai, China, but came to the United States to further his studies in science. In 1948, he invented a magnetic pulse controlling device that vastly increased the storage capacity of computers. He later founded his own company, Wang Laboratories, and became a leader in the development of desktop calculators and word processing systems. In 1988, Wang was elected to the National Inventors Hall of Fame.

Digital computers store information as numbers. Because the electronic circuits of a computer can exist in only one of two states, open or closed, the numbers that are stored can consist of only two digits, 0 or 1. Numbers written using only these two digits are called binary numbers. To find the decimal value of a binary number, you use the digits to write a polynomial in \(2\). For instance, this is how to find the decimal value of the number \(1001101_2\). (The subscript \(2\) indicates that this is a binary number.)

\[
1001101_2 = 1 \times 2^6 + 0 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0
\]
\[
= 1 \times 64 + 0 \times 32 + 0 \times 16 + 1 \times 8 + 1 \times 4 + 0 \times 2 + 1 \times 1
\]
\[
= 64 + 0 + 0 + 8 + 4 + 0 + 1
\]
\[
= 77
\]

Find the decimal value of each binary number.

1. \(1111_2\)  
2. \(10000_2\)  
3. \(11000011_2\)  
4. \(10111001_2\)

Write each decimal number as a binary number.

5. \(8\)  
6. \(11\)  
7. \(29\)  
8. \(117\)

9. The chart at the right shows a set of decimal code numbers that is used widely in storing letters of the alphabet in a computer's memory. Find the code numbers for the letters of your name. Then write the code for your name using binary numbers.
# Enrichment

## Patterns with Powers

Use your calculator, if necessary, to complete each pattern.

<table>
<thead>
<tr>
<th>Column</th>
<th>2^10</th>
<th>2^9</th>
<th>2^8</th>
<th>2^7</th>
<th>2^6</th>
<th>2^5</th>
<th>2^4</th>
<th>2^3</th>
<th>2^2</th>
<th>2^1</th>
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<tr>
<td>Column</td>
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<tr>
<td>Column</td>
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<td>4^9</td>
<td>4^8</td>
<td>4^7</td>
<td>4^6</td>
<td>4^5</td>
<td>4^4</td>
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</table>

Study the patterns for a, b, and c above. Then answer the questions.

1. Describe the pattern of the exponents from the top of each column to the bottom.

2. Describe the pattern of the powers from the top of the column to the bottom.

3. What would you expect the following powers to be?
   \[ 2^0, 5^0, 4^0 \]

4. Refer to Exercise 3. Write a rule. Test it on patterns that you obtain using 22, 25, and 24 as bases.

Study the pattern below. Then answer the questions.

\[ 0^3 = 0 \quad 0^2 = 0 \quad 0^1 = 0 \quad 0^0 = ? \]

5. Why do \( 0^{-1} \), \( 0^{-2} \), and \( 0^{-3} \) not exist?

6. Based upon the pattern, can you determine whether \( 0^0 \) exists?

7. The symbol \( 0^0 \) is called an **indeterminate**, which means that it has no unique value. Thus it does not exist as a unique real number. Why do you think that \( 0^0 \) cannot equal 1?
**8-3 Enrichment**

**Converting Metric Units**

Scientific notation is convenient to use for unit conversions in the metric system.

**Example 1** How many kilometers are there in 4,300,000 meters?

Divide the measure by the number of meters (1000) in one kilometer. Express both numbers in scientific notation.

\[
\frac{4.3 \times 10^6}{1 \times 10^3} = 4.3 \times 10^3
\]

The answer is \(4.3 \times 10^3\) km.

**Example 2** Convert 3700 grams into milligrams.

Multiply by the number of milligrams (1000) in 1 gram.

\[(3.7 \times 10^3)(1 \times 10^3) = 3.7 \times 10^6\]

There are \(3.7 \times 10^6\) mg in 3700 g.

Complete the following. Express each answer in scientific notation.

1. \(250,000\) m = \(\underline{\underline{2.5}} \times 10^5\) km
2. \(375\) km = \(\underline{\underline{3.75}} \times 10^2\) m
3. \(247\) m = \(\underline{\underline{2.47}} \times 10^2\) cm
4. \(5000\) m = \(\underline{\underline{5}} \times 10^3\) mm
5. \(0.0004\) km = \(\underline{\underline{4}} \times 10^{-4}\) m
6. \(0.01\) mm = \(\underline{\underline{1}} \times 10^{-3}\) m
7. \(6000\) m = \(\underline{\underline{6}} \times 10^3\) mm
8. \(340\) cm = \(\underline{\underline{3.4}} \times 10^1\) km
9. \(52,000\) mg = \(\underline{\underline{5.2}} \times 10^4\) g
10. \(420\) kL = \(\underline{\underline{4.2}} \times 10^2\) L

Solve.

11. The planet Mars has a diameter of \(6.76 \times 10^3\) km. What is the diameter of Mars in meters? Express the answer in both scientific and decimal notation.

12. The distance from earth to the sun is \(149,590,000\) km. Light travels \(3.0 \times 10^5\) meters per second. How long does it take light from the sun to reach the earth in minutes? Round to the nearest hundredth.

13. A light-year is the distance that light travels in one year. (See Exercise 12.) How far is a light year in kilometers? Express your answer in scientific notation. Round to the nearest hundredth.
Polynomial Functions

Suppose a linear equation such as $23x + y = 4$ is solved for $y$. Then an equivalent equation, $y = 3x + 4$, is found. Expressed in this way, $y$ is a function of $x$, or $f(x) = 3x + 4$. Notice that the right side of the equation is a binomial of degree 1.

Higher-degree polynomials in $x$ may also form functions. An example is $f(x) = x^3 + 1$, which is a polynomial function of degree 3. You can graph this function using a table of ordered pairs, as shown at the right.

For each of the following polynomial functions, make a table of values for $x$ and $y = f(x)$. Then draw the graph on the grid.

1. $f(x) = 1 - x^2$

2. $f(x) = x^2 - 5$

3. $f(x) = x^2 + 4x - 1$

4. $f(x) = x^3$
Circular Areas and Volumes

Area of Circle

\[ A = \pi r^2 \]

Volume of Cylinder

\[ V = \pi r^2 h \]

Volume of Cone

\[ V = \frac{1}{3} \pi r^2 h \]

Write an algebraic expression for each shaded area. (Recall that the diameter of a circle is twice its radius.)

1.

2.

3.

Write an algebraic expression for the total volume of each figure.

4.

5.

Each figure has a cylindrical hole with a radius of 2 inches and a height of 5 inches. Find each volume.

6.

7.
Figurate Numbers

The numbers below are called pentagonal numbers. They are the numbers of dots or disks that can be arranged as pentagons.

1. Find the product \( \frac{1}{2} n(3n - 1) \).

2. Evaluate the product in Exercise 1 for values of \( n \) from 1 through 4.

3. What do you notice?

4. Find the next six pentagonal numbers.

5. Find the product \( \frac{1}{2} n(n + 1) \).

6. Evaluate the product in Exercise 5 for values of \( n \) from 1 through 5. On another sheet of paper, make drawings to show why these numbers are called the triangular numbers.

7. Find the product \( n(2n - 1) \).

8. Evaluate the product in Exercise 7 for values of \( n \) from 1 through 5. Draw these hexagonal numbers.

9. Find the first 5 square numbers. Also, write the general expression for any square number.

The numbers you have explored above are called the plane figurate numbers because they can be arranged to make geometric figures. You can also create solid figurate numbers.

10. If you pile 10 oranges into a pyramid with a triangle as a base, you get one of the tetrahedral numbers. How many layers are there in the pyramid? How many oranges are there in the bottom layers?

11. Evaluate the expression \( \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n \) for values of \( n \) from 1 through 5 to find the first five tetrahedral numbers.
Pascal’s Triangle

This arrangement of numbers is called Pascal’s Triangle. It was first published in 1665, but was known hundreds of years earlier.

1. Each number in the triangle is found by adding two numbers. What two numbers were added to get the 6 in the 5th row?

2. Describe how to create the 6th row of Pascal’s Triangle.

3. Write the numbers for rows 6 through 10 of the triangle.
   - Row 6:
   - Row 7:
   - Row 8:
   - Row 9:
   - Row 10:

Multiply to find the expanded form of each product.

4. \((a + b)^2\)

5. \((a + b)^3\)

6. \((a + b)^4\)

Now compare the coefficients of the three products in Exercises 4–6 with Pascal’s Triangle.

7. Describe the relationship between the expanded form of \((a + b)^n\) and Pascal’s Triangle.

8. Use Pascal’s Triangle to write the expanded form of \((a + b)^6\).
Sums and Differences of Cubes

Recall the formulas for finding some special products:

Perfect-square trinomials: 

\[(a + b)^2 = a^2 + 2ab + b^2\] or 
\[(a - b)^2 = a^2 - 2ab + b^2\]

Difference of two squares: 
\[(a + b)(a - b) = a^2 - b^2\]

A pattern also exists for finding the cube of a sum \((a + b)^3\).

1. Find the product of \((a + b)(a + b)(a + b)\).

2. Use the pattern from Exercise 1 to evaluate \((x + 2)^3\).

3. Based on your answer to Exercise 1, predict the pattern for the cube of a difference \((a - b)^3\).

4. Find the product of \((a - b)(a - b)(a - b)\) and compare it to your answer for Exercise 3.

5. Use the pattern from Exercise 4 to evaluate \((x + 4)^3\).

Find each product.

6. \((x + 6)^3\) 
7. \((x - 10)^3\)

8. \((3x - y)^3\) 
9. \((2x - y)^3\)

10. \((4x + 3y)^3\) 
11. \((5x + 2)^3\)
Finding the GCF by Euclid’s Algorithm

Finding the greatest common factor of two large numbers can take a long time using prime factorizations. This method can be avoided by using Euclid’s Algorithm as shown in the following example.

Example

Find the GCF of 209 and 532.

Divide the greater number, 532, by the lesser, 209.

\[
\begin{array}{c|c|c}
\text{209} & \text{532} \\
\hline
\text{418} & \text{1} \\
\text{114} & \underline{209} \\
\end{array}
\]

Divide the remainder into the divisor above.
Repeat this process until the remainder is zero. The last nonzero remainder is the GCF.

\[
\begin{array}{c|c|c}
\text{114} & \text{1} \\
\text{95} & \underline{114} \\
\text{19} & \underline{95} \\
\text{0} & \underline{19} \\
\end{array}
\]

The divisor, 19, is the GCF of 209 and 532.

Suppose the GCF of two numbers is found to be 1. Then the numbers are said to be relatively prime.

Find the GCF of each group of numbers by using Euclid’s Algorithm.

1. 187; 578
2. 1802; 106

3. 161; 943
4. 215; 1849

5. 1325; 3498
6. 3484; 5963

7. 33,583; 4257
8. 453; 484

9. 95; 209; 589
10. 518; 407; 851

11. \(17a^2x^2z^2; 1615axz^2\)
12. \(752cf^3; 893c^3f^3\)

13. \(979r^2s^2; 495rs^3, 154r^3s^3\)
14. \(360x^5y^7; 328xy, 568x^3y^3\)
Perfect, Excessive, Defective, and Amicable Numbers

A **perfect number** is the sum of all of its factors except itself. Here is an example:

\[ 28 = 1 + 2 + 4 + 7 + 14 \]

There are very few perfect numbers. Most numbers are either excessive or defective.

An **excessive number** is greater than the sum of all of its factors except itself.

A **defective number** is less than this sum.

Two numbers are **amicable** if the sum of the factors of the first number, except for the number itself, equals the second number, and vice versa.

**Solve each problem.**

1. Write the perfect numbers between 0 and 31.

2. Write the excessive numbers between 0 and 31.

3. Write the defective numbers between 0 and 31.

4. Show that 8128 is a perfect number.

5. The sum of the reciprocals of all the factors of a perfect number (including the number itself) equals 2. Show that this is true for the first two perfect numbers.

6. More than 1000 pairs of amicable numbers have been found. One member of the first pair is 220. Find the other member.

7. One member of the second pair of amicable numbers is 2620. Find the other member.

8. The Greek mathematician Euclid proved that the expression \( 2^n - 1(2^n - 1) \) equals a perfect number if the expression inside the parentheses is prime. Use Euclid's expression with \( n \) equal to 5 to find the third perfect number.
Puzzling Primes

A prime number has only two factors, itself and 1. The number 6 is not prime because it has 2 and 3 as factors; 5 and 7 are prime. The number 1 is not considered to be prime.

1. Use a calculator to help you find the 25 prime numbers less than 100.

Prime numbers have interested mathematicians for centuries. They have tried to find expressions that will give all the prime numbers, or only prime numbers. In the 1700s, Euler discovered that the trinomial \( x^2 + x + 41 \) will yield prime numbers for values of \( x \) from 0 through 39.

2. Find the prime numbers generated by Euler’s formula for \( x \) from 0 through 7.

3. Show that the trinomial \( x^2 + x + 31 \) will not give prime numbers for very many values of \( x \).

4. Find the largest prime number generated by Euler’s formula.

Goldbach’s Conjecture is that every nonzero even number greater than 2 can be written as the sum of two primes. No one has ever proved that this is always true, but no one has found a counterexample, either.

5. Show that Goldbach’s Conjecture is true for the first 5 even numbers greater than 2.

6. Give a way that someone could disprove Goldbach’s Conjecture.
Area Models for Quadratic Trinomials

After you have factored a quadratic trinomial, you can use the factors to draw geometric models of the trinomial.

\[ x^2 + 5x - 6 = (x - 1)(x + 6) \]

To draw a rectangular model, the value 2 was used for \( x \) so that the shorter side would have a length of 1. Then the drawing was done in centimeters. So, the area of the rectangle is \( x^2 + 5x - 6 \).

To draw a right triangle model, recall that the area of a triangle is one-half the base times the height. So, one of the sides must be twice as long as the shorter side of the rectangular model.

\[ x^2 + 5x - 6 = (x - 1)(x + 6) \]
\[ = \frac{1}{2} (2x - 2)(x + 6) \]

The area of the right triangle is also \( x^2 + 5x - 6 \).

Factor each trinomial. Then follow the directions to draw each model of the trinomial.

1. \( x^2 + 2x - 3 \)  Use \( x = 2 \). Draw a rectangle in centimeters.
2. \( 3x^2 + 5x - 2 \)  Use \( x = 1 \). Draw a rectangle in centimeters.

3. \( x^2 - 4x + 3 \)  Use \( x = 4 \). Draw two different right triangles in centimeters.

4. \( 9x^2 - 9x + 2 \)  Use \( x = 2 \). Draw two different right triangles.
   Use 0.5 centimeter for each unit.
Factoring Trinomials of Fourth Degree

Some trinomials of the form \(a^4 + a^2b^2 + b^4\) can be written as the difference of two squares and then factored.

**Example**

Factor \(4x^4 - 37x^2y^2 + 9y^4\).

**Step 1** Find the square roots of the first and last terms.
\[
\sqrt{4x^4} = 2x^2 \quad \sqrt{9y^4} = 3y^2
\]

**Step 2** Find twice the product of the square roots.
\[
2(2x^2)(3y^2) = 12x^2y^2
\]

**Step 3** Separate the middle term into two parts. One part is either your answer to Step 2 or its opposite. The other part should be the opposite of a perfect square.
\[
-37x^2y^2 = -12x^2y^2 - 25x^2y^2
\]

**Step 4** Rewrite the trinomial as the difference of two squares and then factor.
\[
4x^4 - 37x^2y^2 + 9y^4 = (4x^4 - 12x^2y^2 + 9y^4) - 25x^2y^2
\]
\[
= (2x^2 - 3y^2)^2 - 25x^2y^2
\]
\[
= [(2x^2 - 3y^2) + 5xy][(2x^2 - 3y^2) - 5xy]
\]
\[
= (2x^2 + 5xy - 3y^2)(2x^2 - 5xy - 3y^2)
\]

Factor each trinomial.

1. \(x^4 + x^2y^2 + y^4\)  
2. \(x^4 + x^2 + 1\)

3. \(9a^4 - 15a^2 + 1\)  
4. \(16a^4 - 17a^2 + 1\)

5. \(4a^4 - 13a^2 + 1\)  
6. \(9a^4 + 26a^2b^2 + 25b^4\)

7. \(4x^4 - 21x^2y^2 + 9y^4\)  
8. \(4a^4 - 29a^2c^2 + 25c^4\)
Squaring Numbers: A Shortcut

A shortcut helps you to square a positive two-digit number ending in 5. The method is developed using the idea that a two-digit number may be expressed as $10t + u$. Suppose $u = 5$.

$$(10t + 5)^2 = (10t + 5)(10t + 5)$$

$$= 100t^2 + 50t + 50t + 25$$

$$= 100t^2 + 100t + 25$$

$$(10t + 5)^2 = 100t(t + 1) + 25$$

In words, this formula says that the square of a two-digit number has $t(t + 1)$ in the hundreds place. Then 2 is the tens digit and 5 is the units digit.

Example

Using the formula for $(10t + 5)^2$, find $85^2$.

$$85^2 = 100 \cdot 8 \cdot (8 + 1) + 25$$

$$= 7200 + 25$$

$$= 7225$$

Shortcut: First think $8 \cdot 9 = 72$. Then write 25.

Thus, to square a number, such as 85, you can write the product of the tens digit and the next consecutive integer $t + 1$. Then write 25.

Find each of the following using the shortcut.

1. $15^2$
2. $25^2$
3. $35^2$
4. $45^2$
5. $55^2$
6. $65^2$

Solve each problem.

7. What is the tens digit in the square of 95?

8. What are the first two digits in the square of 75?

9. Any three-digit number can be written as $100a + 10b + c$. Square this expression to show that if the last digit of a three-digit number is 5 then the last two digits of the square of the number are 2 and 5.
Translating Quadratic Graphs

When a figure is moved to a new position without undergoing any rotation, then the figure is said to have been translated to that position.

The graph of a quadratic equation in the form \( y = (x - b)^2 + c \) is a translation of the graph of \( y = x^2 \).

Start with \( y = x^2 \).

Slide to the right 4 units.
\[ y = (x - 4)^2 \]

Then slide up 3 units.
\[ y = (x - 4)^2 + 3 \]

These equations have the form \( y = x^2 + c \). Graph each equation.

1. \( y = x^2 + 1 \)
2. \( y = x^2 + 2 \)
3. \( y = x^2 - 2 \)

These equations have the form \( y = (x - b)^2 \). Graph each equation.

4. \( y = (x - 1)^2 \)
5. \( y = (x - 3)^2 \)
6. \( y = (x + 2)^2 \)
Odd Numbers and Parabolas

The solid parabola and the dashed stair-step graph are related. The parabola intersects the stair steps at their inside corners.

Use the figure for Exercises 1–3.

1. What is the equation of the parabola?

2. Describe the horizontal sections of the stair-step graph.

3. Describe the vertical sections of the stair-step graph.

Use the second figure for Exercises 4–6.

4. What is the equation of the parabola?

5. Describe the horizontal sections of the stair steps.

6. Describe the vertical sections.

7. How does the graph of \( y = \frac{1}{2}x^2 \) relate to the sequence of numbers \( \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots \)?

8. Complete this conclusion. To graph a parabola with the equation \( y = ax^2 \), start at the vertex. Then go over 1 and up \( a \); over 1 and up \( 3a \);
Parabolas Through Three Given Points

If you know two points on a straight line, you can find the equation of the line. To find the equation of a parabola, you need three points on the curve.

Here is how to approximate an equation of the parabola through the points (0, -2), (3, 0), and (5, 2).

Use the general equation $y = ax^2 + bx + c$. By substituting the given values for $x$ and $y$, you get three equations.

$(0, -2): -2 = c$
$(3, 0): 0 = 9a + 3b + c$
$(5, 2): 2 = 25a + 5b + c$

First, substitute $-2$ for $c$ in the second and third equations. Then solve those two equations as you would any system of two equations. Multiply the second equation by 5 and the third equation by $\frac{1}{3}$.

$0 = 9a + 3b - 2$ Multiply by 5.
$0 = 45a + 15b - 10$

$0 = 25a + 5b - 2$ Multiply by $\frac{1}{3}$.
$-6 = -75a - 15b + 6$

To find $b$, substitute $\frac{1}{15}$ for $a$ in either the second or third equation.

$0 = 9\left(\frac{1}{15}\right) + 3b - 2$

$b = \frac{7}{15}$

The equation of a parabola through the three points is

$y = \frac{1}{15}x^2 + \frac{7}{15}x - 2$.

Find the equation of a parabola through each set of three points.

1. (1, 5), (0, 6), (2, 3)       2. (−5, 0), (0, 0), (8, 100)
3. (4, −4), (0, 1), (3, −2)       4. (1, 3), (6, 0), (0, 0)
5. (2, 2), (5, −3), (0, −1)       6. (0, 4), (4, 0), (−4, 4)
Mechanical Constructions of Parabolas

A given line and a point determine a parabola. Here is one way to construct the curve.

Use a right triangle $ABC$ (or a stiff piece of rectangular cardboard).

Place one leg of the triangle on the given line $d$.

Fasten one end of a string with length $BC$ at the given point $F$ and the other end to the triangle at point $B$.

Put the tip of a pencil at point $P$ and keep the string tight.

As you move the triangle along the line $d$, the point of your pencil will trace a parabola.

Draw the parabola determined by line $d$ and point $F$.

1. ____________________________ $d$    2. ____________________________

3. ____________________________    4. ____________________________

5. Use your drawings to complete this conclusion. The greater the distance of point $F$ from line $d$, ____________________________
10-5 Enrichment

Writing Expressions of Area in Factored Form

Write an expression in factored form for the area $A$ of the shaded region in each figure below.

1. \[ x \times y \times y \times x \]

2. \[ r \times r \]

3. \[ r \times b \]

4. \[ r \times 4r \]

5. \[ R \]

6. \[ a \times b \times 2a \]

7. \[ 4x \times 4x \times 4x \times 2x \]

8. \[ r \times r \times r \]
Curious Circles

Two circles can be arranged in four ways: one circle can be inside the other, they can be separate, they can overlap, or they can coincide.

In how many ways can a given number of circles be either separate or inside each other? (The situations in which the circles overlap or coincide are not counted here.)

For 3 circles, there are 4 different possibilities.

1 2 3 4

Solve each problem. Make drawings to show your answers.

1. Show the different ways in which 2 circles can be separate or inside each other. How many ways are there?

2. Show the different ways for 4 circles. How many ways are there?

3. Use your answer for Exercise 2 to show that the number of ways for 5 circles is at least 18.

4. Find the number of ways for 5 circles. Show your drawings on a separate sheet of paper.
Convergence, Divergence, and Limits

Imagine that a runner runs a mile from point A to point B. But, this is not an ordinary race! In the first minute, he runs one-half mile, reaching point C. In the next minute, he covers one-half the remaining distance, or \(\frac{1}{4}\) mile, reaching point D. In the next minute he covers one-half the remaining distance, or \(\frac{1}{8}\) mile, reaching point E.

\[
\begin{array}{c|c|c|c|c}
\text{0 mile} & \frac{1}{2} \text{ mile} & \frac{3}{4} \text{ mile} & \frac{7}{8} \text{ mile} & 1 \text{ mile} \\
\hline
A & C & D & E & B
\end{array}
\]

(3:00 P.M.) (3:01 P.M.) (3:02 P.M.) (3:03 P.M.) (?)

In this strange race, the runner approaches closer and closer to point B, but never gets there. However close he is to B, there is still some distance remaining, and in the next minute he can cover only half of that distance. This race can be modeled by the infinite sequence

\[
\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots
\]

The terms of the sequence get closer and closer to 1. An infinite sequence that gets arbitrarily close to some number is said to converge to that number. The number is the limit of the sequence.

Not all infinite sequences converge. Those that do not are called divergent.

Write C if the sequence converges and D if it diverges. If the sequence converges, make a reasonable guess for its limit.

1. 2, 4, 6, 8, 10, ...
2. 0, 3, 0, 3, 0, 3, ...
3. 1, \(\frac{1}{2}\), \(\frac{1}{3}\), \(\frac{1}{4}\), \(\frac{1}{5}\), ...
4. 0.9, 0.99, 0.999, 0.9999, ...
5. \(-5, 5, -5, 5, -5, 5, ...
6. 0.1, 0.2, 0.3, 0.4, ...
7. \(2 \frac{1}{4}, 2 \frac{3}{4}, 2 \frac{7}{8}, 2 \frac{15}{16}, ...
8. 6, 5 \frac{1}{2}, 5 \frac{1}{3}, 5 \frac{1}{4}, 5 \frac{1}{5}, ...
9. 1, 4, 9, 16, 25, ...
10. \(-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{5}, ...

11. Create one convergent sequence and one divergent sequence. Give the limit for your convergent sequence.
Squares and Square Roots From a Graph

The graph of \( y = x^2 \) can be used to find the squares and square roots of numbers.

To find the square of 3, locate 3 on the \( x \)-axis. Then find its corresponding value on the \( y \)-axis.

The arrows show that \( 3^2 = 9 \).

To find the square root of 4, first locate 4 on the \( y \)-axis. Then find its corresponding value on the \( x \)-axis. Following the arrows on the graph, you can see that \( \sqrt{4} = 2 \).

A small part of the graph at \( y = x^2 \) is shown below. A 1:10 ratio for unit length on the \( y \)-axis to unit length on the \( x \)-axis is used.

Find \( \sqrt{11} \).

The arrows show that \( \sqrt{11} = 3.3 \) to the nearest tenth.

Use the graph above to find each of the following to the nearest whole number.

1. \( 1.5^2 \)  
2. \( 2.7^2 \)  
3. \( 0.9^2 \)  
4. \( 3.6^2 \)  
5. \( 4.2^2 \)  
6. \( 3.9^2 \)

Use the graph above to find each of the following to the nearest tenth.

7. \( \sqrt{15} \)  
8. \( \sqrt{8} \)  
9. \( \sqrt{3} \)  
10. \( \sqrt{5} \)  
11. \( \sqrt{14} \)  
12. \( \sqrt{17} \)
The Wheel of Theodorus

The Greek mathematicians were intrigued by problems of representing different numbers and expressions using geometric constructions.

Theodorus, a Greek philosopher who lived about 425 B.C., is said to have discovered a way to construct the sequence \( \sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots \).

The beginning of his construction is shown. You start with an isosceles right triangle with sides 1 unit long.

Use the figure above. Write each length as a radical expression in simplest form.

1. line segment \( AO \)  
2. line segment \( BO \)

3. line segment \( CO \)  
4. line segment \( DO \)

5. Describe how each new triangle is added to the figure.

6. The length of the hypotenuse of the first triangle is \( \sqrt{2} \). For the second triangle, the length is \( \sqrt{3} \). Write an expression for the length of the hypotenuse of the \( n \)th triangle.

7. Show that the method of construction will always produce the next number in the sequence. (Hint: Find an expression for the hypotenuse of the \( (n + 1) \)th triangle.)

8. In the space below, construct a Wheel of Theodorus. Start with a line segment 1 centimeter long. When does the Wheel start to overlap?
11-3 Enrichment

Special Polynomial Products

Sometimes the product of two polynomials can be found readily with the use of one of the special products of binomials.

For example, you can find the square of a trinomial by recalling the square of a binomial.

Example 1

Find \((x + y + z)^2\).

\[
(a + b)^2 = a^2 + 2 \cdot a \cdot b + b^2
\]

\[
(x + y + z)^2 = (x + y)^2 + 2(x + y)z + z^2
= x^2 + 2xy + y^2 + 2xz + 2yz + z^2
\]

Example 2

Find \((3t + x + 1)(3t - x - 1)\).

(Hint: \((3t + x + 1)(3t - x - 1)\) is the product of a sum \(3t + (x + 1)\) and a difference \(3t - (x + 1)\).)

\[
(3t + x + 1)(3t - x - 1) = [3t + (x + 1)][3t - (x + 1)]
= 9t^2 - (x + 1)^2
= 9t^2 - x^2 - 2x - 1
\]

Use a special product of binomials to find each product.

1. \((x + y - z)^2\)  
2. \((r + s + 5)^2\)

3. \((b - 3 + d)^2\)  
4. \((k - m - 2)^2\)

5. \((x + 1 + 2b)(x + 1 - 2b)\)  
6. \((y - 2 + x)(y - 2 - x)\)

7. \((5 + b - x)(5 + b + x)\)  
8. \((j - 5 - f)(j + 5 + f)\)

9. \([(x + y) + (z + w)][(x + y) - (z + w)]\)  
10. \((2a + 1 + 3b - c)(2a + 1 - 3b + c)\)
Pythagorean Triples

Recall the Pythagorean theorem:
In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.

The integers 3, 4, and 5 satisfy the Pythagorean theorem and can be the lengths of the sides of a right triangle.

Furthermore, for any positive integer \( n \), the numbers \( 3n \), \( 4n \), and \( 5n \) satisfy the Pythagorean theorem.

If three numbers satisfy the Pythagorean theorem, they are called a Pythagorean triple. Here is an easy way to find other Pythagorean triples.

The numbers \( a \), \( b \), and \( c \) are a Pythagorean triple if \( a = m^2 - n^2 \), \( b = 2mn \), and \( c = m^2 + n^2 \), where \( m \) and \( n \) are relatively prime positive integers and \( m > n \).

**Example**

Choose \( m = 5 \) and \( n = 2 \).

\[
\begin{align*}
    a &= m^2 - n^2 = 5^2 - 2^2 = 25 - 4 = 21 \\
    b &= 2mn = 2(5)(2) = 20 \\
    c &= m^2 + n^2 = 5^2 + 2^2 = 25 + 4 = 29 \\
    \text{Check} & \quad 20^2 + 21^2 = 29^2 \\
    & \quad 400 + 441 = 841
\end{align*}
\]

Use the following values of \( m \) and \( n \) to find Pythagorean triples.

1. \( m = 3 \) and \( n = 2 \)  
2. \( m = 4 \) and \( n = 1 \)  
3. \( m = 5 \) and \( n = 3 \)

4. \( m = 6 \) and \( n = 5 \)  
5. \( m = 10 \) and \( n = 7 \)  
6. \( m = 8 \) and \( n = 5 \)
A Space-Saving Method

Two arrangements for cookies on a 32 cm by 40 cm cookie sheet are shown at the right. The cookies have 8-cm diameters after they are baked. The centers of the cookies are on the vertices of squares in the top arrangement. In the other, the centers are on the vertices of equilateral triangles. Which arrangement is more economical? The triangle arrangement is more economical, because it contains one more cookie.

In the square arrangement, rows are placed every 8 cm. At what intervals are rows placed in the triangle arrangement?

Look at the right triangle labeled \(a\), \(b\), and \(c\). A leg \(a\) of the triangle is the radius of a cookie, or 4 cm. The hypotenuse \(c\) is the sum of two radii, or 8 cm. Use the Pythagorean theorem to find \(b\), the interval of the rows.

\[
\begin{align*}
  c^2 &= a^2 + b^2 \\
  8^2 &= 4^2 + b^2 \\
  64 - 16 &= b^2 \\
  \sqrt{48} &= b \\
  4\sqrt{3} &= b \\
  b &= 4\sqrt{3} \approx 6.93
\end{align*}
\]

The rows are placed approximately every 6.93 cm.

Solve each problem.

1. Suppose cookies with 10-cm diameters are arranged in the triangular pattern shown above. What is the interval \(b\) of the rows?

2. Find the diameter of a cookie if the rows are placed in the triangular pattern every \(3\sqrt{3}\) cm.

3. Describe other practical applications in which this kind of triangular pattern can be used to economize on space.
A Curious Construction

Many mathematicians have been interested in ways to construct the number \( \pi \). Here is one such geometric construction.

In the drawing, triangles \( ABC \) and \( ADE \) are right triangles. The length of \( AD \) equals the length of \( AC \) and \( FB \) is parallel to \( EG \).

The length of \( BG \) gives a decimal approximation of the fractional part of \( \pi \) to six decimal places.

Follow the steps to find the length of \( BG \). Round to seven decimal places.

1. Use the length of \( BC \) and the Pythagorean Theorem to find the length of \( AC \).

2. Find the length of \( AD \).

3. Use the length of \( AD \) and the Pythagorean Theorem to find the length of \( AE \).

4. The sides of the similar triangles \( FED \) and \( DEA \) are in proportion. So, \( \frac{FE}{0.5} = \frac{0.5}{AE} \). Find the length of \( FE \).

5. Find the length of \( AF \).

6. The sides of the similar triangles \( AFB \) and \( AEG \) are in proportion. So, \( \frac{AF}{AE} = \frac{AB}{AG} \). Find the length of \( AG \).

7. Now, find the length of \( BG \).

8. The value of \( \pi \) to seven decimal places is 3.1415927. Compare the fractional part of \( \pi \) with the length of \( BG \).
Modern Art

The painting below, aptly titled, “Right Triangles,” was painted by that well-known artist Two-loose La-Rectangle. Using the information below, find the dimensions of this masterpiece. (Hint: The triangle that includes $\angle F$ is isosceles.)

\[ \tan A = \frac{2}{5} \quad \tan B = \frac{4}{5} \quad \tan C = \frac{1}{2} \]
\[ \tan D = \frac{1}{2} \quad \tan E = 4 \quad \tan F = 1 \]
\[ \tan G = \frac{1}{3} \]

1. What is the length of the painting?

2. What is the width of the painting?
Direct or Indirect Variation

Fill in each table below. Then write inversely, or directly to complete each conclusion.

1. | l | 2 | 4 | 8 | 16 | 32 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>w</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

For a set of rectangles with a width of 4, the area varies ____________ as the length.

2. | Hours | 2 | 4 | 5 | 6 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Speed</td>
<td>55</td>
<td>55</td>
<td>55</td>
<td>55</td>
</tr>
<tr>
<td>Distance</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For a car traveling at 55 mi/h, the distance covered varies ____________ as the hours driven.

3. | Oat Bran | $\frac{1}{3}$ cup | $\frac{2}{3}$ cup | 1 cup |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Water</td>
<td>1 cup</td>
<td>2 cups</td>
<td>3 cups</td>
</tr>
</tbody>
</table>

The number of servings of oat bran varies ____________ as the number of cups of oat bran.

4. | Hours of Work | 128 | 128 | 128 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>People Working</td>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>Hours per Person</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

A job requires 128 hours of work. The number of hours each person works varies ____________ as the number of people working.

5. | Miles | 100 | 100 | 100 | 100 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate</td>
<td>20</td>
<td>25</td>
<td>50</td>
<td>100</td>
</tr>
<tr>
<td>Hours</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For a 100-mile car trip, the time the trip takes varies ____________ as the average rate of speed the car travels.

6. | b | 3 | 4 | 5 | 6 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>h</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>A</td>
<td>15</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For a set of right triangles with a height of 10, the area varies ____________ as the base.

Use the table at the right.

7. $x$ varies ____________ as $y$.

8. $z$ varies ____________ as $y$.

9. $x$ varies ____________ as $z$. 

Use the table at the right.

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>z</td>
<td>60</td>
<td>40</td>
<td>30</td>
<td>24</td>
<td>20</td>
</tr>
</tbody>
</table>
12-2 Enrichment

Rational Exponents

You have developed the following properties of powers when \( a \) is a positive real number and \( m \) and \( n \) are integers.

\[
\begin{align*}
\text{For integers } m, n: \\
am^m \cdot a^n &= a^{m+n} \\
(a^m)^n &= a^{mn} \\
\frac{a^m}{a^n} &= a^{m-n} \\
(a^m)^n &= a^{mn} \\
a^0 &= 1 \\
a^{-m} &= \frac{1}{a^m}
\end{align*}
\]

Exponents need not be restricted to integers. We can define rational exponents so that operations involving them will be governed by the properties for integer exponents.

\[
\begin{align*}
\left(\frac{1}{a^2}\right)^2 &= \left(\frac{1}{a}\right)^2 = a^2 \\
\left(\frac{1}{a^3}\right)^3 &= \left(\frac{1}{a}\right)^3 = a^3 \\
\left(\frac{1}{a^n}\right)^n &= \left(\frac{1}{a}\right)^n = a^n \\
\left(\frac{1}{a^m}\right)^n &= \left(\frac{1}{a}\right)^{mn} = \left(\sqrt[n]{a}ight)^m
\end{align*}
\]

Therefore, \( a^m = \sqrt[n]{a^m} \) or \( a^m = \sqrt[n]{a}^m \).

**Example 1** Write \( \sqrt[4]{a^3} \) in exponential form.

\[
\sqrt[4]{a^3} = a^{\frac{3}{4}}
\]

**Example 2** Write \( a^{\frac{2}{5}} \) in radical form.

\[
a^{\frac{2}{5}} = \sqrt[5]{a^2}
\]

**Example 3** Find \( \frac{a^{\frac{2}{3}}}{a^{\frac{1}{2}}} \).

\[
\frac{a^{\frac{2}{3}}}{a^{\frac{1}{2}}} = a^{\frac{2}{3} - \frac{1}{2}} = a^{\frac{4}{6} - \frac{3}{6}} = a^{\frac{1}{6}}
\]

Write each expression in radical form.

1. \( b^{\frac{3}{2}} \) 
   2. \( 3c^{\frac{1}{2}} \) 
   3. \( (3c)^{\frac{1}{2}} \)

Write each expression in exponential form.

4. \( \sqrt[3]{b^4} \) 
   5. \( \sqrt[4]{a^3} \) 
   6. \( 2 \cdot \sqrt[3]{b^2} \)

Perform the operation indicated. Answers should show positive exponents only.

7. \( \left(\frac{a^2b^3}{b^2}\right)^{\frac{1}{2}} \) 
   8. \( \frac{-8a^{\frac{3}{2}}}{2a^2} \) 
   9. \( \left(\frac{b^{\frac{1}{2}}}{b^{-\frac{3}{2}}}\right)^3 \)

10. \( \sqrt[3]{a^3} \cdot \sqrt[2]{a} \) 
    11. \( (a^2b^{\frac{1}{3}})^{\frac{1}{2}} \) 
    12. \( -2a^2b^{\frac{1}{2}}(5a^2b^{-\frac{2}{3}}) \)
Continued Fractions

The following is an example of a continued fraction. By starting at the bottom you can simplify the expression to a rational number.

\[
3 + \frac{4}{1 + \frac{6}{7}} = 3 + \frac{4}{1 + \frac{13}{7}}
\]

\[
= 3 + \frac{28}{13} \text{ or } \frac{67}{13}
\]

**Example**

Express \(\frac{48}{19}\) as a continued fraction.

\[
\frac{48}{19} = 2 + \frac{10}{19}
\]

Notice that the numerator of the last fraction must be equal to 1 before the process stops.

\[
= 2 + \frac{1}{\frac{19}{10}}
\]

\[
= 2 + \frac{1}{1 + \frac{9}{10}}
\]

\[
= 2 + \frac{1}{1 + \frac{1}{\frac{10}{9}}}
\]

\[
= 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{9}}}
\]

Write each continued fraction as a rational number.

1. \(6 + \frac{1}{1 + \frac{1}{3 + \frac{1}{3}}}\)

2. \(5 + \frac{7}{2 + \frac{3}{4 + \frac{2}{3}}}\)

Write each rational number as a continued fraction.

3. \(\frac{97}{17}\)

4. \(\frac{22}{64}\)
Division by Zero?

You may remember being told, “division by zero is not possible” or “division by zero is undefined” or “we never divide by zero.” Have you wondered why this is so? Consider the two equations below.

\[
\frac{5}{0} = n \quad \frac{0}{0} = m
\]

Because multiplication is the inverse of division, these equations lead to the following.

\[
0 \cdot n = 5 \quad 0 \cdot m = 0
\]

There is no number that will make the first equation true. Any number at all will satisfy the second equation.

For each expression, give the values that must be excluded from the replacement set in order to prevent division by zero.

1. \(\frac{x + 1}{x - 1}\)
2. \(\frac{2(x + 1)}{2x - 1}\)
3. \(\frac{(x + 1)(x - 1)}{(x + 2)(x - 2)}\)
4. \(\frac{x + y + 3}{(3x - 1)(3y - 1)}\)
5. \(\frac{x^2 + y^2 + z^2}{2xyz}\)
6. \(\frac{(x + y)^2}{x - y}\)

Many demonstrations or “proofs” that lead to impossible results include a step involving division by zero. Explain what is wrong with each “proof” below.

7. \(0 \cdot 1 = 0\) and \(0 \cdot 2 = 0\).
   Therefore, \(\frac{0}{0} = 1\) and \(\frac{0}{0} = 2\).
   Therefore, \(1 = 2\).

8. Assume that \(a = b\).
   Then \(ab = a^2\).
   Therefore, \(ab - b^2 = a^2 - b^2\).
   Next it is shown that \(a^2 - b^2 = (a + b)(a - b)\).
   \((a + b)(a - b) = (a + b)a - (a + b)b\)
   \(= a^2 + ba - ab - b^2\)
   \(= a^2 + 0 - b^2\)
   \(= a^2 - b^2\)
   Therefore, \(ab - b^2 = (a + b)(a - b)\).
   Also, \(b(a - b) = ba - b^2 = ab - b^2\).
   Therefore, \(b(a - b) = (a + b)(a - b)\).
   Therefore, \(b = a + b\).
   Therefore, \(b = 2b\).
   Therefore, \(1 = 2\).

NAME ______________________________________________ DATE ____________ PERIOD _____

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12-5 Enrichment

Synthetic Division

You can divide a polynomial such as $3x^3 - 4x^2 - 3x - 2$ by a binomial such as $x - 3$ by a process called synthetic division. Compare the process with long division in the following explanation.

Example

Divide $(3x^3 - 4x^2 - 3x - 2)$ by $(x - 3)$ using synthetic division.

1. Show the coefficients of the terms in descending order.
2. The divisor is $x - 3$. Since 3 is to be subtracted, write 3 in the corner.
3. Bring down the first coefficient, 3.
4. Multiply. $3 \cdot 3 = 9$
5. Add. $-4 + 9 = 5$
6. Multiply. $3 \cdot 5 = 15$
7. Add. $-3 + 15 = 12$
8. Multiply. $3 \cdot 12 = 36$
9. Add. $-2 + 36 = 34$

Check Use long division.

$$
\begin{array}{c}
3x^2 + 5x + 12,
\end{array}
$$

Divide by using synthetic division. Check your result using long division.

1. $(x^3 + 6x^2 + 3x + 1) \div (x - 2)$
2. $(x^3 - 3x^2 - 6x - 20) \div (x - 5)$
3. $(2x^3 - 5x + 1) \div (x + 1)$
4. $(3x^3 - 7x^2 + 4) \div (x - 2)$
5. $(x^3 + 2x^2 - x + 4) \div (x + 3)$
6. $(x^3 + 4x^2 - 3x - 11) \div (x - 4)$
**12-6 Enrichment**

**Sum and Difference of Any Two Like Powers**

The sum of any two like powers can be written $a^n + b^n$, where $n$ is a positive integer. The difference of like powers is $a^n - b^n$. Under what conditions are these expressions exactly divisible by $(a + b)$ or $(a - b)$? The answer depends on whether $n$ is an odd or even number.

Use long division to find the following quotients. *(Hint: Write $a^3 + b^3$ as $a^3 + 0a^2 + 0a + b^3$.) Is the numerator exactly divisible by the denominator? Write yes or no.*

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$\frac{a^3 + b^3}{a + b}$</td>
<td>2.</td>
<td>$\frac{a^3 + b^3}{a - b}$</td>
<td>3.</td>
</tr>
<tr>
<td>5.</td>
<td>$\frac{a^4 + b^4}{a + b}$</td>
<td>6.</td>
<td>$\frac{a^4 + b^4}{a - b}$</td>
<td>7.</td>
</tr>
<tr>
<td>9.</td>
<td>$\frac{a^5 + b^5}{a + b}$</td>
<td>10.</td>
<td>$\frac{a^5 + b^5}{a - b}$</td>
<td>11.</td>
</tr>
</tbody>
</table>

13. Use the words *odd* and *even* to complete these two statements.
   a. $a^n + b^n$ is divisible by $a + b$ if $n$ is __________, and by neither $a + b$ nor $a - b$ if $n$ is __________.
   b. $a^n - b^n$ is divisible by $a - b$ if $n$ is __________, and by both $a + b$ and $a - b$ if $n$ is __________.

14. Describe the signs of the terms of the quotients when the divisor is $a - b$.

15. Describe the signs of the terms of the quotient when the divisor is $a + b$. 

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Graphing Circles by Completing Squares

One use for completing the square is to graph circles. The general equation for a circle with center at the origin and radius \( r \) is \( x^2 + y^2 = r^2 \). An equation represents a circle if it can be transformed into the sum of two squares.

\[
(x^2 - 6x + 9) + (y^2 + 4y + 4) = 3 + 9 + 4 \\
(x - 3)^2 + (y + 2)^2 = 4^2
\]

Notice that the center of the circle is at the point (3, -2).

Transform each equation into the sum of two squares. Then graph the circle represented by the equation. Use the coordinate plane provided at the bottom of the page.

1. \( x^2 - 14x + y^2 + 6y + 49 = 0 \)  
2. \( x^2 + y^2 - 8y - 9 = 0 \)

3. \( x^2 + 10x + y^2 + 21 = 0 \)  
4. \( x^2 + y^2 + 10y + 16 = 0 \)

5. \( x^2 - 30x + y^2 + 209 = 0 \)  
6. \( x^2 - 18x + y^2 - 12y + 116 = 0 \)

7. \( x^2 + 30x + y^2 - 4y + 193 = 0 \)  
8. \( x^2 + 38x + y^2 - 12y + 393 = 0 \)
Complex Fractions

Complex fractions are really not complicated. Remember that a fraction can be interpreted as dividing the numerator by the denominator.

\[
\frac{\frac{2}{3}}{\frac{5}{7}} = \frac{2}{3} \div \frac{5}{7} = \frac{2 \cdot 7}{3 \cdot 5} = \frac{14}{15}
\]

Let \(a, b, c,\) and \(d\) be numbers, with \(b \neq 0, c \neq 0,\) and \(d \neq 0.\)

\[
\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}
\]

Notice the pattern: denominator of the answer \((bc)\) ←− \(\frac{a}{b}\) \(\rightarrow\) numerator of the answer \((ad)\) \[\rightarrow \frac{a}{b} \div \frac{c}{d} \rightarrow \frac{ad}{bc}\]

**Example 1**

Simplify \(\frac{\frac{5x}{4}}{\frac{x + 2}{3}}.\)

\[
\frac{\frac{5x}{4}}{\frac{x + 2}{3}} = \frac{5x \cdot 3}{4(x + 2)} = \frac{15x}{4x + 8}
\]

**Example 2**

Simplify \(\frac{\frac{x}{2} + 4}{3x - 2}.\)

\[
\frac{\frac{x}{2} + 4}{3x - 2} = \frac{\frac{x + 8}{2}}{3x - 2} = \frac{(x + 8)(1)}{2(3x - 2)} = \frac{x + 8}{6x - 4}
\]

Simplify each complex fraction.

1. \(\frac{\frac{2x}{5}}{\frac{y}{6}}\)
2. \(\frac{\frac{5x}{3}}{x}\)
3. \(\frac{\frac{x - 3}{2x + 1}}{4}\)
4. \(\frac{x^2 + \frac{1}{3}}{4x + \frac{1}{3}}\)
5. \(\frac{1 - \frac{1}{x}}{\frac{2}{5} - 1}\)
6. \(\frac{\frac{x + 2x^{-2}}{2 + \frac{x}{3}}}{\frac{x + 2}{x - 2} + \frac{1}{x + 2 + \frac{1}{x}}}\)
Winning Distances

In 1999, Hicham El Guerrouj set a world record for the mile run with a time of 3:43.13 (3 h 43 min 13 s). In 1954, Roger Bannister ran the first mile under 4 minutes at 3:59.4. Had they run those times in the same race, how far in front of Bannister would El Guerrouj have been at the finish?

Use \( \frac{d}{t} = r \). Since 3 min 43.13 s = 223.13 s, and 3 min 59.4 s = 239.4 s,

El Guerrouj’s rate was \( \frac{5280 \text{ ft}}{223.13 \text{ s}} \) and Bannister’s rate was \( \frac{5280 \text{ ft}}{239.4 \text{ s}} \).

<table>
<thead>
<tr>
<th>r</th>
<th>t</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>El Guerrouj</td>
<td>223.13</td>
<td>5280</td>
</tr>
<tr>
<td>Bannister</td>
<td>239.4</td>
<td>5280 \times 223.13 or 4921.2 feet</td>
</tr>
</tbody>
</table>

Therefore, when El Guerrouj hit the tape, he would be \( 5280 - 4921.2 = 358.8 \) feet, ahead of Bannister. Let’s see whether we can develop a formula for this type of problem.

Let \( D \) = the distance raced,
\( W \) = the winner’s time,
and \( L \) = the loser’s time.

Following the same pattern, you obtain the results shown in the table at the right.

The winning distance will be \( D - \frac{DW}{L} \).

1. Show that the expression for the winning distance is equivalent to \( \frac{D(L - W)}{L} \).

<table>
<thead>
<tr>
<th>r</th>
<th>t</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>Winner</td>
<td>( \frac{D}{W} )</td>
<td>W</td>
</tr>
<tr>
<td>Loser</td>
<td>( \frac{D}{L} )</td>
<td>W</td>
</tr>
</tbody>
</table>

Use the formula winning distance = \( \frac{D(L - W)}{L} \) to find the winning distance to the nearest tenth for each of the following Olympic races.

2. women’s 400 meter relay: Canada 48.4 s (1928);
   East Germany 41.6 s (1980)

3. men’s 200 meter freestyle swimming: Yevgeny Sadovyi 1 min 46.70 s (1992);
   Michael Gross 1 min 47.44 s (1984)

4. men’s 50,000 meter walk: Vyacheslav Ivanenko 3 h 38 min 29 s (1988);
   Hartwig Gauter 3 h 49 min 24 s (1980)

5. women’s 400 meter freestyle relay: United States 3 min 39.29 s (1996);
   East Germany 3 min 42.71 s (1980)
**Geometric Vanishing Acts**

Puzzles of this type use a “trick” drawing. It appears that rearranging the pieces of each figure causes one or more squares to disappear.

Make figures of your own on graph paper. Then explain the “trick” in each puzzle.

1. The rectangle has an area of 65 square units, but the square has an area of only 64 square units.

2. The square has an area of 64 square units, but the rectangle has an area of only 63 square units.

3. The square has an area of 64 units, but the new figure has an area of only 63 units.

4. Rearranging the square on the left causes a 2-unit “hole” to appear.
**Geometric Series**

The terms of this polynomial form a geometric series.

\[ a + ar + ar^2 + ar^3 + ar^4 \]

The first term is the constant \( a \). Then each term after that is found by multiplying by a constant multiplier \( r \).

**Use the equation** \( S = a + ar + ar^2 + ar^3 + ar^4 \) for Exercises 1–3.

1. Multiply each side of the equation by \( r \).

2. Subtract the original equation from your result in Exercise 1.

3. Solve the result from Exercise 2 for the variable \( S \).

**Use the polynomial** \( a + ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} \) for Exercises 4–8.

4. Write the 10th term of the polynomial.

5. If \( a = 5 \) and \( r = 2 \), what is the 8th term?

6. Follow the steps in Exercises 1–3 to write a formula for the sum of this polynomial \( S \).

7. If the 3rd term is 20 and the 6th term is 160, write a division expression to solve for \( r^3 \) and then find \( r \). Then solve \( ar^2 = 20 \) for \( a \) and find the value of the first six terms of the polynomial.

8. Find the sum of the first six terms of the geometric series that begins 3, 6, 12, 24, \ldots. First write the values for \( a \) and \( r \).
Surface Area of Solid Figures

Many solid objects are formed by rectangles and squares. A box is an example.

The dimensions of the box shown at the right are represented by letters. The length of the base is $\ell$ units, its width is $w$ units, and the height of the box is $h$ units.

Suppose the box is cut on the seams so that it can be spread out on a flattened surface as shown at the right. The area of this figure is the surface area of the box. Find a formula for the surface area of the box.

There are 6 rectangles in the figure. The surface area is the sum of the areas of the 6 rectangles.

$$S = hw + h\ell + \ell w + h\ell + hw + \ell w$$

$$S = 2\ell w + 2h\ell + 2hw$$

Find the surface area of a box with the given dimensions.

1. $\ell = 14$ cm, $w = 8$ cm, $h = 2$ cm

2. $\ell = 40$ cm, $w = 30$ cm, $h = 25$ cm

3. $\ell = x$ cm, $w = (x - 3)$ cm, $h = (x + 3)$ cm

4. $\ell = (s + 9)$ cm, $w = (s - 9)$ cm, $h = (s + 9)$ cm

5. The surface area of a box is 142 square centimeters. The length of the base is 2 centimeters longer than its width. The height of the box is 2 centimeters less than the width of the base. Find the dimensions of the box.

6. Write an expression that represents the surface area of the figure shown at the right. Include the surface area of the base.
**Standard Deviation**

The most commonly used measure of variation is called the **standard deviation**. It shows how far the data are from their mean. You can find the standard deviation using the steps given below.

a. Find the mean of the data.
b. Find the difference between each value and the mean.
c. Square each difference.
d. Find the mean of the squared differences.
e. Find the square root of the mean found in Step d. The result is the standard deviation.

**Example**

Calculate the standard deviation of the test scores 82, 71, 63, 78, and 66.

\[
\text{mean of the data (} m \text{)} = \frac{82 + 71 + 63 + 78 + 66}{5} = \frac{360}{5} = 72
\]

<table>
<thead>
<tr>
<th>(x)</th>
<th>(x - m)</th>
<th>((x - m)^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>82</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>71</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>63</td>
<td>-9</td>
<td>81</td>
</tr>
<tr>
<td>78</td>
<td>6</td>
<td>36</td>
</tr>
<tr>
<td>66</td>
<td>-6</td>
<td>36</td>
</tr>
</tbody>
</table>

\[
\text{mean of the squared differences} = \frac{100 + 1 + 81 + 36 + 36}{5} = \frac{254}{5} = 50.8
\]

\[
\text{standard deviation} = \sqrt{50.8} \approx 7.13
\]

**Use the test scores 94, 48, 83, 61, and 74 to complete Exercises 1–3.**

1. Find the mean of the scores.

2. Show that the standard deviation of the scores is about 16.2.

3. Which had fewer variations, the test scores listed above or the test scores in the example?


13-5 Enrichment

**Which Scientist Is Which?**

Just as you can solve some problems by systematically listing all the possibilities, you can solve other problems by systematically eliminating possibilities. The following puzzle will give you some practice in using this problem-solving strategy.

Berthel Carmichael, Subrahmanyan Chandrasekhar, Jack Murata, Tsung Dao Lee, and Severe Ochoa are an agricultural chemist, an astrophysicist, a biochemist, a physicist, and a research mathematician, but not necessarily in that order. The following are some facts about them.

1. The biochemist is neither Murata nor Chandrasekhar.
2. Ochoa was born before the astrophysicist.
3. The name of the research mathematician begins with the letter C.
4. Neither Carmichael nor Murata is the astrophysicist or the physicist.
5. Lee was born after both the biochemist and the astrophysicist.

Who is the agricultural chemist?

To find out which scientist is which, read the clues carefully. Record an × in the chart when you know that a match is not possible. When all but one possibility in a row or column has been eliminated, you’ve found a match. Mark it with a ✔.

The name of the agricultural chemist is _____________________________.

The table at the right lists five other prominent scientists, the years of their birth, and their fields of specialization. Use this information to create a puzzle similar to the one above.

<table>
<thead>
<tr>
<th>Homi Jehangir Bhabha (1909)</th>
<th>theoretical physicist</th>
</tr>
</thead>
<tbody>
<tr>
<td>Takao Doi (1954)</td>
<td>aeronautical engineer</td>
</tr>
<tr>
<td>Mae Jemison (1957)</td>
<td>astronaut</td>
</tr>
<tr>
<td>Cho Hao Li (1913)</td>
<td>biochemist</td>
</tr>
<tr>
<td>César Milstein (1927)</td>
<td>molecular biologist</td>
</tr>
</tbody>
</table>
**Pascal’s Triangle**

**Pascal’s Triangle** is a pattern of numbers used at many levels of mathematics. It is named for Blaise Pascal, a seventeenth-century French mathematician who discovered several applications of the pattern. However, records of the triangle have been traced as far back as twelfth-century China and Persia. In the year 1303, the Chinese mathematician Zhū-Shijié wrote *The Precious Mirror of the Four Elements*, in which he described how the triangle could be used to solve polynomial equations. The figure at the right is adapted from the original Chinese manuscript. In the figure, some circles are empty while others contain Chinese symbols.

At the right, a portion of Pascal’s Triangle is shown using Hindu-Arabic numerals.

```
  1
1 1
  1
```

The triangle expresses a relationship between numbers that you can discover by comparing the Chinese version and the Hindu-Arabic version.

1. What Chinese symbol corresponds to the Hindu-Arabic numeral 1?

2. Fill in the outermost circles in the Chinese version of Pascal’s Triangle.

3. What Chinese symbol corresponds to the Hindu-Arabic numeral 4?

4. What Chinese symbol corresponds to the Hindu-Arabic numeral 10?

5. Based upon your investigation so far, fill in as many of the missing numbers as you can in both the Chinese and Hindu-Arabic versions of Pascal’s Triangle.

6. Pascal’s Triangle is *symmetric* about an imaginary vertical line that separates the left and right halves of the triangle. Use this fact to fill in more missing numbers in the triangles.

7. Each row of the triangle is generated from the row above by using a simple rule. Find the rule. Then fill in the remaining entries in both triangles.
**Latin Squares**

In designing a statistical experiment, it is important to try to randomize the variables. For example, suppose 4 different motor oils are being compared to see which give the best gasoline mileage. An experimenter might then choose 4 different drivers and four different cars. To test-drive all the possible combinations, the experimenter would need 64 test-drives.

To reduce the number of test drives, a statistician might use an arrangement called a **Latin Square**.

For this example, the four motor oils are labeled A, B, C, and D and are arranged as shown. Each oil must appear exactly one time in each row and column of the square.

The drivers are labeled D₁, D₂, D₃, and D₄; the cars are labeled C₁, C₂, C₃, and C₄.

Now, the number of test-drives is just 16, one for each cell of the Latin Square.

Create two 4-by-4 Latin Squares that are different from the example.

1.  

2.  

Make three different 3-by-3 Latin Squares.

3.  

4.  

5.
Conditional Probability

The probability of an event given the occurrence of another event is called **conditional probability**. The conditional probability of event $A$ given event $B$ is denoted $P(A|B)$.

**Example**
Suppose a pair of number cubes is rolled. It is known that the sum is greater than seven. Find the probability that the number cubes match.

There are 15 sums greater than seven and there are 36 possible pairs altogether.

$$P(B) = \frac{15}{36}$$

The conditional probability is

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

$$= \frac{3}{36} \text{ or } \frac{1}{5}$$

The conditional probability is $\frac{1}{5}$.

A card is drawn from a standard deck of 52 cards and is found to be red. Given that event, find each of the following probabilities.

1. $P(\text{heart})$
2. $P(\text{ace})$
3. $P(\text{face card})$
4. $P(\text{jack or ten})$
5. $P(\text{six of spades})$
6. $P(\text{six of hearts})$

A sports survey taken at Stirers High School shows that 48% of the respondents liked soccer, 66% liked basketball, and 38% liked hockey. Also, 30% liked soccer and basketball, 22% liked basketball and hockey, and 28% liked soccer and hockey. Finally, 12% liked all three sports.

7. Find the probability that Meg likes soccer if she likes basketball.

8. Find the probability that Juan likes basketball if he likes soccer.

9. Find the probability that Mieko likes hockey if she likes basketball.

10. Find the probability that Greg likes hockey if he likes soccer.
Golden Rectangles

A golden rectangle has the property that its sides satisfy the following proportion.

\[
\frac{a + b}{a} = \frac{a}{b}
\]

Two quadratic equations can be written from the proportion. These are sometimes called golden quadratic equations.

1. In the proportion, let \( a = 1 \). Use cross-multiplication to write a quadratic equation.

2. Solve the equation in Exercise 1 for \( b \).

3. In the proportion, let \( b = 1 \). Write a quadratic equation in \( a \).

4. Solve the equation in Exercise 3 for \( a \).

5. Explain why \( \frac{1}{2}(\sqrt{5} + 1) \) and \( \frac{1}{2}(\sqrt{5} - 1) \) are called golden ratios.

Another property of golden rectangles is that a square drawn inside a golden rectangle creates another, smaller golden rectangle.

In the design at the right, opposite vertices of each square have been connected with quarters of circles.

For example, the arc from point \( B \) to point \( C \) is created by putting the point of a compass at point \( A \). The radius of the arc is the length \( BA \).

6. On a separate sheet of paper, draw a larger version of the design. Start with a golden rectangle with a long side of 10 inches.
The Work Problem and Similar Right Triangles

“The work problem” has been included in algebra textbooks for a very long time. In older books, the people in the problem always seemed to be digging ditches.

If Olivia can dig a ditch in \( x \) hours and George can dig the same ditch in \( y \) hours, how long will it take them to dig the ditch if they work together?

You have learned a way to solve this type of problem using rational equations. It can also be solved using a geometric model that uses two overlapping right triangles.

In the drawing, the length \( x \) is Olivia’s time. The length \( y \) is George’s time. The answer to the problem is the length of the segment \( z \). The distance \( m + n \) can be any convenient length.

Solve each problem.

1. Solve the work problem for \( x = 6 \) and \( y = 3 \) by drawing a diagram and measuring.

2. Confirm your solution to Exercise 1 by writing and solving a rational equation.

3. On a separate sheet of paper, create a word problem to go with the values \( x = 6 \) and \( y = 3 \).

4. On a separate sheet of paper, solve this problem with a diagram. Use centimeters and measure to the nearest tenth. Olivia can wash a car in 3 hours. George can wash a car in 4 hours. How long will it take them working together to wash one car?

5. Triangles that have the same shape are called similar triangles. You may have learned that corresponding sides of similar triangles form equal ratios. Using the drawing at the top of the page, you can thus conclude that Equations A and B below are true. Use the equations to prove the formula for the work problem.

\[
\frac{z}{x} = \frac{n}{m + n} \quad \frac{z}{y} = \frac{m}{m + n} \quad \frac{1}{x} + \frac{1}{y} = \frac{1}{z}
\]