Key Concepts

Equations as Functions

Objective Teach students to determine whether an equation represents a function.

Note to the Teacher *This lesson introduces the vertical line test for determining whether a relation is a function. Make sure students realize that for each value of x, the vertical line passes through no more than one point on the graph of a function. Be sure to link this idea to the formal definition of function. In a function, for each domain value, there is one and only one range value.*

Equations that Define Relations

Suppose we have an equation in two variables x and y. Let's consider

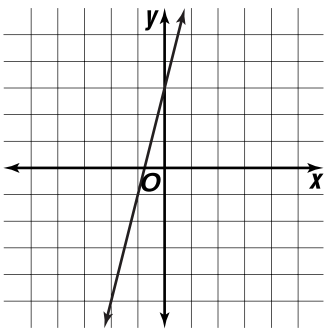
$$y = 4x + 3$$

and

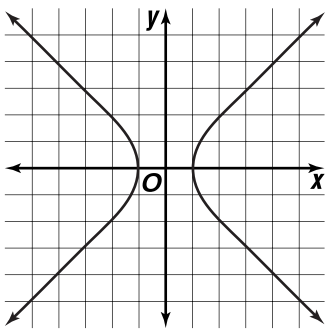
$$x^2 - y^2 = 1.$$

For any equation, the solution set is the set of ordered pairs (x, y) for which the equation holds true when the values are substituted for x and y. Since a set of ordered pairs is a relation, we can conclude that a solution set is a relation. In this way, equations represent relations.

A relation that can be represented by an equation can easily be visualized by graphing the equation. The graphs of $y = 4x + 3$ and $x^2 - y^2 = 1$ are shown.



$y = 4x + 3$



$x^2 - y^2 = 1$

When is a Relation a Function?

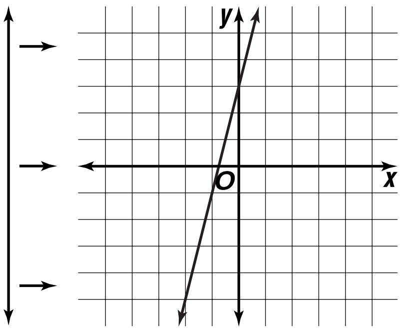
The set of first coordinates in a relation is called the **domain** of the relation. The set of second coordinates is called the **range**. Remember that a relation is a **function** if each element of the domain is paired with exactly one element in the range.

When we have a relation defined by an equation, we can graph the equation and then visually determine whether or not it is a function by using the **vertical line test**.

Key Idea	For a relation defined by an equation, the relation is a function if every vertical line intersects the graph of the equation in at most one point.

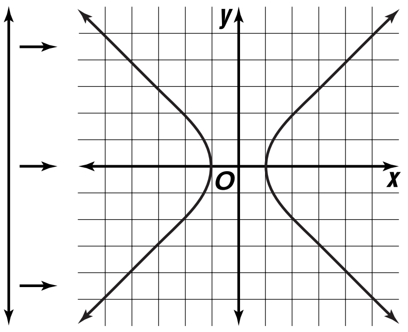
Consider the graph of $y = 4x + 3$. Using the vertical line test, we find that the equation does define a function.

As the vertical line moves to the right across the graph, it intersects only one point at a time.



Consider the graph of $x^2 - y^2 = 1$. Using the vertical line test, we find that the equation does not define a function.

As the vertical line moves to the right across the graph, it usually intersects the graph in two points. Sometimes it does not intersect the graph at all.



Recognizing Functions Algebraically

Another way to recognize when an equation defines a function is to solve for y in terms of x.

Key Idea	Any equation which has y on one side and a single formula involving x on the other side defines a function.

The following equations are examples of functions.

$$y = x + 3 \qquad y = x^2 + 4 \qquad y = \frac{1}{2}x - 1 \qquad y = x^2 + 3x + 2$$

Some equations are not functions. Consider $x^2 - y^2 = 1$. If $x^2 - y^2 = 1$ is solved for y, the result is $y = \pm\sqrt{x^2 - 1}$. The symbol $\pm$ indicates that there are two formulas on the right side of the equation for y. Hence, $x^2 - y^2 = 1$ does not represent a function.

If an equation is not written in the form $y =$ (formula in x), the equation may still determine a function. Consider the following example.

In $4x - 2y = 10$, y is not isolated on the left side. However, we can solve the equation for y.

$$4x - 2y = 10$$

$$4x - 2y - 4x = 10 - 4x \qquad \textit{Subtract 4x from each side.}$$

$$-2y = -4x + 10$$

$$\frac{-2y}{-2} = \frac{-4x + 10}{-2} \qquad \textit{Divide each side by } -2.$$

$$y = 2x - 5$$

This is now in the form $y =$ (formula in x) and gives a function.

Key Idea	If, in an equation, we can solve for y as a single formula in terms of x, then that equation represents a function.

Functions as Rules

Functions can also be thought of as rules that take an input value of x and produce an output value of y. We often give these rules names such as f, g, h, etc. We can then define a function by a formula in x.

The rule $f(x) = x^2 + 5$ represents a function. We can think of it as a relation by taking the function to be the set of all ordered pairs of the form $(x, f(x)) = (x, x^2 + 5)$. It is also the solution set of $y = x^2 + 5$.

Often we are given an equation in which y is not given as a formula in x, but we can solve for y in terms of x to get a rule defining the function.

In $y + x = x^2$, we can solve for y by subtracting x from each side to get

$$y = x^2 - x.$$

This equation now gives a function. If we name the function f, then the function is defined by

$$f(x) = x^2 - x.$$