Key Concepts

Negative Exponents

Objective Introduce the notation of negative and zero exponents.

Note to the Teacher *The concept of exponentiation as repeated multiplication cannot be extended to negative exponents. Therefore, this lesson may be conceptually difficult for many students.*

Negative Numbers and Zero in the Exponent

First, review the rule for dividing powers that have the same base.

Key Idea	When we divide a power of a by another smaller power of a, the result is a power of a, where the exponent is the difference of the exponents of the two dividends. In symbols, $\frac{a^b}{a^c} = a^{b-c}$ when $b > c$.

Point out that only exponentiation with positive integers has been defined. We have not discussed exponentiation with zero or with negative numbers. In the formula

$$\frac{a^b}{a^c} = a^{b-c},$$

ask students what happens if we no longer insist that b be greater than c. What definition would this suggest to us for zero and negative exponents?

If we applied the formula for $b = c$, we would get

$$\frac{a^b}{a^b} = a^{b-b} = a^0.$$

On the other hand, $\frac{a^b}{a^b} = 1$. So, the formula suggests that $a^0 = 1$.

Key Idea	For any nonzero number a, $a^0 = 1$.

What does the equation shown above suggest about negative numbers?

Let's consider a^{-1}. According to the formula, we have

$$\frac{a}{a^2} = \frac{a^1}{a^2} = a^{1-2} = a^{-1}.$$

On the other hand,

$$\frac{a}{a^2} = \frac{a}{a \cdot a} = \frac{a}{a} \cdot \frac{1}{a} = 1 \cdot \frac{1}{a} = \frac{1}{a}.$$

So the formula suggests that $a^{-1} = \frac{1}{a}$. In general, we define a^{-n} as $\frac{1}{a^n}$.

Key Idea	For any nonzero number a and any integer n, $a^{-n} = \frac{1}{a^n}$.

Note to the Teacher *Students may be confused by the way the formula is used to generate a definition for negative powers. Point out that it is used as a guide to suggest what a definition should be, but that once we make the definition, we can work with these exponents in exactly the same way as we did with positive exponents. Emphasize that with this definition, these exponents satisfy all the laws of exponents previously studied.*

The following formula shows what happens when b is less than c.

$$\frac{a^b}{a^c} = \frac{\overbrace{a \cdot a \cdot \ldots \cdot a}^{b \text{ factors}}}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{c \text{ factors}}} = \frac{\overbrace{a \cdot a \cdot \ldots \cdot a}^{b \text{ factors}}}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{b \text{ factors}}} \cdot \frac{1}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{b - c \text{ factors}}} = 1 \cdot \frac{1}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{b - c \text{ factors}}} = \frac{1}{a^{b-c}}$$

Note to the Teacher *At this point, have students evaluate various negative powers of numbers so that they can become familiar with the notation. It may also be helpful to have students work with a calculator to further familiarize themselves with negative exponents.*

We conclude with a real-world example of negative exponents.

Suppose a certain bacterium doubles each week. The table below shows how many bacteria will be present after 5 weeks if there was an initial population of a million bacteria.

Number of Weeks	Number of Bacteria (millions)
0	1
1	2
2	4
3	8
4	16
5	32

The table was made using only the fact that the population doubles each week. Notice that the number of bacteria is always a power of 2 times one million.

Number of Weeks	Number of Bacteria (millions)
0	2^0 or 1
1	2^1 or 2
2	2^2 or 4
3	2^3 or 8
4	2^4 or 16
5	2^5 or 32

Note to the Teacher *Now is a good time to ask students to find a formula for the number of bacteria present after t weeks.*

Suppose the bacteria have been growing for t weeks. The following formula describes the number of bacteria present after any number of weeks.

$$\text{number of bacteria after } t \text{ weeks} = 2^t \text{ million bacteria}$$

By substituting a value for t, we can find the number of bacteria present after any number of weeks. For instance, after 10 weeks, we will have

$$2^t \text{ million bacteria} = 2^{10} \text{ million or 1024 million bacteria.}$$

Using negative exponents, we can also tell how many bacteria were present at some time in the past. Suppose we wanted to know how many bacteria were present 3 weeks ago. We can think of 3 weeks ago as “negative three weeks” from now. Thus, we substitute *t* with −3.

$$
\begin{aligned}
2^{-3} \text{ million bacteria} &= \frac{1}{8} \text{ million bacteria} \\
&= \frac{1}{8} \cdot 1{,}000{,}000 \text{ bacteria} \\
&= \frac{1{,}000{,}000}{8} \text{ bacteria} \\
&= 125{,}000 \text{ bacteria}
\end{aligned}
$$

So by using the same formula but with negative exponents, we can find how many bacteria were present at a time in the past as well as how many will be present in the future.

End of Lesson