Key Concepts

Probability of Compound Events

Objective Introduce students to the concept of compound independent events, and teach students the formula for evaluating the probability of a compound event in terms of the probabilities of the individual events.

Note to the Teacher *A key idea in this lesson is that of independent events. Be sure to stress this idea and the formula that says the probability of two independent events occurring simultaneously is the product of the probabilities of the individual events.*

Independent and Dependent Events

Remember that in a probability experiment, an **event** is just a collection of outcomes. For instance, if we are rolling a die, an event might be that we roll an even number. This event would include the outcomes {2, 4, 6}. In the same way, if we are drawing a card from a standard deck of playing cards, an event might be that we draw a king. This event would include the outcomes {king of hearts, king of diamonds, king of clubs, king of spades}. Sometimes we are interested in two different events and the likelihood that they will both occur at the same time.

Example 1 **Suppose we roll a fair die and toss a fair coin at the same time. What is the probability that we will roll an even number and toss heads?**

Solution First, we list all the possible outcomes.

(1, heads), (2, heads), (3, heads), (4, heads),
(5, heads), (6, heads), (1, tails), (2, tails),
(3, tails), (4, tails), (5, tails), (6, tails)

There are 12 possible outcomes, each of which is equally likely, since the die and the coin are both fair. Among these outcomes, there are three that are in the event, namely

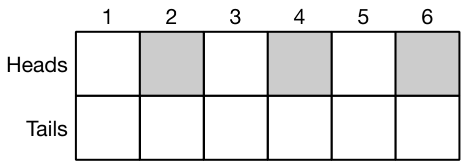
(2, heads), (4, heads), (6, heads).

Since the outcomes are equally likely, the probability of this event is $\frac{3}{12}$ or $\frac{1}{4}$.

Let's take another look at the experiment in Example 1. Suppose we only consider one of the events, namely that we roll an even number. There are a total of six possible outcomes, namely {1, 2, 3, 4, 5, 6}, and three of them, {2, 4, 6}, are in the event. So the probability of rolling an even number is $\frac{3}{6}$ or $\frac{1}{2}$. In the same way, the probability that we get heads on the coin toss is $\frac{1}{2}$. Now we make an important observation.

The probability of the two events occurring simultaneously is the product of the probabilities of the individual events, since
$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

We can picture this graphically, by drawing a rectangle, in which one side length is divided into 6 equal parts (for the six different possible outcomes for rolling the die) and the other side length is divided into 2 equal parts (for the two different possible outcomes for tossing the coin).



The shaded region in the figure represents the simultaneous occurrence of the two events, an even number rolled on the die and heads tossed on the coin. This model is similar to those used to represent multiplying two fractions. We can see that the shaded region is $\frac{1}{4}$ of the total rectangle.

Key Idea	If in a probability experiment, we have two independent events *A* and *B*, then the probability of both events occurring simultaneously is $P(A) \cdot P(B)$.

However, this kind of calculation cannot be done for all simultaneous events, as shown in Example 2.

Example 2 **Suppose we have a hat containing 2 red balls and 2 white balls. We draw a ball from the hat, and then draw a second ball without replacing the first one. What is the probability that we draw two red balls?**

Solution We need to find the probability of drawing a red ball on the first draw and a red ball on the second draw when no replacement occurs. To describe all the outcomes, imagine numbering the balls from 1 to 4, with the two red balls being 1 and 2 and the two white balls being 3 and 4. The number of possible outcomes is now the number of permutations that result when choosing two numbers from this list of four numbers, so the number of possible outcomes is $P(4, 2) = 4 \times 3$ or 12. Each of these outcomes is equally likely. Of these 12 outcomes, there are two in which both balls are red, namely the outcome (ball 1, ball 2) and (ball 2, ball 1). So the probability is $\frac{2}{12}$ or $\frac{1}{6}$.

Stress that in this situation, because the first ball is *not* replaced, the probability is *not* $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

The difference between the events in the two previous examples is that in Example 1 the two events are *independent*, meaning that the result of one of the events has no effect on the outcome of the other. In Example 1, the roll of the die has no effect on the outcome of the coin toss, and vice versa. In Example 2, the drawing of the first ball does have an effect on the outcome of the second draw because the ball is not replaced. We say that the events in Example 2 are *dependent*. We will discuss dependent events at length after doing one more example.

Note to the Teacher *The idea of independent events is a very important one. It is a good idea to discuss examples like the previous ones to make sure students understand the distinction between independent and dependent events.*

Example 3 **Suppose we have two well-shuffled decks of playing cards and we draw one card from each deck. What is the probability of drawing two kings?**

Solution Since there are 4 kings in a standard deck of 52 playing cards, the likelihood of drawing a king from one deck is $\frac{4}{52}$ or $\frac{1}{13}$. In this case, we are asked to find the probability that we draw two kings, one from each of two decks. These events are independent, because the drawing of a card from one deck has no effect on the drawing from the other deck. Therefore, the probability of drawing two kings is $\frac{1}{13} \times \frac{1}{13}$ or $\frac{1}{169}$.

Point out that if we had drawn both cards from one deck without replacing the first card in the deck, then the two drawings would have been dependent, since there would be one less card for the second drawing. If we replaced the first card in the deck and shuffled the deck, then the two drawings from a single deck would again be independent, since the first drawing would have no effect on the second one.

Dependent Events and Conditional Probability

Even when events are dependent, we can often determine the probability that both events will occur simultaneously.

Refer back to Example 2 where the experiment involved selecting two balls from a hat without replacement. The probability of selecting a red ball on the first draw is $\frac{1}{2}$, since there are 2 red balls and 2 white balls. Once a red ball has been chosen, we need to determine the probability that we will select a red ball on the second draw. This is referred as the *conditional probability* that we select a red ball on the second draw, given that we have already selected a red ball on the first draw. Point out that having drawn a red ball on the first draw, the hat now contains 1 red ball and 2 white balls. So, the probability of drawing a second red ball from the hat now is $\frac{1}{3}$. Therefore, the probability of drawing 2 red balls, without replacement, can be computed as

$$\begin{pmatrix}\text{probability of}\\ \text{selecting a red ball}\\ \text{on the first draw}\end{pmatrix} \times \begin{pmatrix}\text{probability of selecting a red}\\ \text{ball on the second draw,}\\ \text{given that a red ball was}\\ \text{selected on the first draw}\end{pmatrix} \text{ or}$$

$$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}.$$

This result agrees with the answer we found when we computed the probability using permutations in Example 2.

We will write $P(B, A)$ for the conditional probability of B happening given that we know that A has happened.

We can now make the following general statement using conditional probability. It is important that students understand that events A and B can be either dependent *or* independent events.

Key Idea	Suppose we have two events, A and B. Then the probability that they both occur is $P(A) \cdot P(B, A)$. When events A and B are independent, then $P(B, A) = P(B)$, and the probability that both events occur is again $P(A) \cdot P(B)$.

Point out that the second sentence of the Key Idea is simply a restatement of the Key Idea from earlier in this lesson.