Key Concepts

Adding and Subtracting Polynomials

Objective Teach students to add and subtract polynomials.

Note to the Teacher *This lesson is not difficult conceptually, but it is important that students get sufficient practice in addition and subtraction of polynomials.*

Introduce the notion of addition of polynomials by presenting the following example.

Example 1 **Anne and Joe both make and sell square tablecloths. They use different pricing schemes to determine the prices of their tablecloths. Anne charges \$1.00 for every square foot of area plus \$2.00 for every foot of perimeter of her tablecloths. Joe charges \$1.50 for every square foot of area plus \$1.00 for every foot of perimeter of his tablecloths. Suppose you buy identical square tablecloths of length x, one from Anne and one from Joe. In terms of x, how much will you pay in all for the two tablecloths?**

Solution We first write down how much we pay for the tablecloth produced by Anne. The area of the square tablecloth is the square of the length of the side, or x^2. The perimeter is 4 times the length of each side, since the square has four identical sides. This means the perimeter is $4x$. Since Anne charges \$1.00 for every square foot of area and \$2.00 for every foot of perimeter, we can write the cost of Anne's tablecloth as follows.

$$\begin{aligned}\$1 \times (\text{area}) + \$2 \times (\text{perimeter}) &= \$1 \cdot x^2 + \$2 \cdot 4x \\ &= x^2 + 8x \text{ dollars}\end{aligned}$$

Notice that the cost is a polynomial.

Now let's do the same thing for the tablecloth Joe has made. Since he charges \$1.50 for every square foot of area and \$1.00 for every foot of perimeter, the cost of his tablecloth is as follows.

$$\begin{aligned}\$1.50 \times (\text{area}) + \$1 \times (\text{perimeter}) &= \$1.50 \cdot x^2 + \$1 \cdot 4x \\ &= 1.5x^2 + 4x \text{ dollars}\end{aligned}$$

This cost is also a polynomial.

The total price we will pay for the two tablecloths is found by adding these two polynomials together. So we can write

$$(x^2 + 8x) + (1.5x^2 + 4x) \text{ dollars}$$

for the total cost of the two tablecloths. We can look at this expression and see that it can be simplified, since there are two x^2 terms and two x terms.

$$\begin{aligned}(x^2 + 8x) + (1.5x^2 + 4x) &= x^2 + 1.5x^2 + 8x + 4x \\ &= (1 + 1.5)x^2 + (8 + 4)x \\ &= 2.5x^2 + 12x \text{ dollars}\end{aligned}$$

Notice that this way of writing the result is shorter than the original sum, since each power of x occurs only once.

How Do We Add Two Polynomials Together?

The following example shows how to add polynomials.

Example 2 **Add $x^3 + x + 1$ and $3x^3 + x^2 + 2x$.**

Solution To add the polynomials $x^3 + x + 1$ and $3x^3 + x^2 + 2x$ together, we will first write them as an addition sentence.

$$(x^3 + x + 1) + (3x^3 + x^2 + 2x)$$

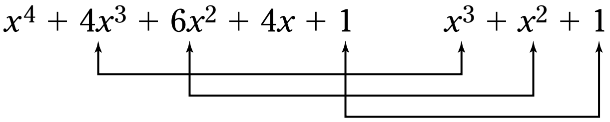
Notice that there is an x^3-term and an x-term in each polynomial, so we can combine terms in the same way we did in Example 1. Begin by rearranging the terms being added.

$$\begin{aligned}&(x^3 + x + 1) + (3x^3 + x^2 + 2x) \\ &\quad = (x^3 + 3x^3) + x^2 + (x + 2x) + 1 \\ &\quad = 4x^3 + x^2 + 3x + 1\end{aligned}$$

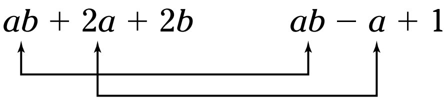
Now introduce the concept of like terms to better explain the process shown in Example 2.

Like terms are monomials that have the same variable to the same power. So, $-4x^3$ and $7x^3$ are like terms because they both contain the variable x raised to the third power. Constants are also considered to be like terms.

Suppose we have the two polynomials $x^4 + 4x^3 + 6x^2 + 4x + 1$ and $x^3 + x^2 + 1$. The two polynomials have several pairs of like terms, as shown in the graphic below, where pairs of like terms are connected by arrows.



Now suppose we have the two polynomials $ab + 2a + 2b$ and $ab - a + 1$. In this case, we have two pairs of like terms.



Whenever we have like terms, we can collect them and add them together to get a single term. In the first example above, we can add $4x^3$ and x^3 together to get $5x^3$. In the same way, we can add $6x^2$ and x^2 together to get $7x^2$, and finally add 1 and 1 to get 2. Doing this, we can write the sum of the two polynomials as

$$x^4 + 5x^3 + 7x^2 + 4x + 2.$$

In the second example above, we can combine the like terms: $ab + ab = 2ab$, and $2a + (-a) = [2 + (-1)]a = a$. Doing this, we write the sum of the polynomials as

$$2ab + a + 2b + 1.$$

Key Idea	Sums of polynomials can be simplified by combining all like terms. In the same way, differences of polynomials can be simplified by combining all pairs of like terms.

Subtracting Polynomials

Ask students to recall the procedure for subtracting integers. Remind them that subtracting an integer was the same as *adding the opposite* of that integer. That is, $5 - 12$ is the same as $5 + (-12)$ and $-4 - 3$ is the same as $-4 + (-3)$. The same technique can be applied to the subtraction of polynomials.

Example 3 **Find $(x^3 - 3x^2 + 3x - 1) - (x^2 + 2x + 1)$.**

Solution First rewrite the subtraction as the addition of the opposite of the terms of the second polynomial.

$$(x^3 - 3x^2 + 3x - 1) - (x^2 + 2x + 1)$$
$$= (x^3 - 3x^2 + 3x - 1) + (-x^2 - 2x - 1)$$

Identify all the pairs of like terms in the sum.

$$(x^3 - 3x^2 + 3x - 1) \;+\; (-x^2 - 2x - 1)$$

When we combine the pairs of like terms, we get

$$(x^3 - 3x^2 + 3x - 1) + (-x^2 - 2x - 1)$$
$$= x^3 + [-3 + (-1)]x^2 + [3 + (-2)]x + [-1 + (-1)]$$
$$= x^3 + (-4)x^2 + 1x + (-2) \text{ or } x^3 - 4x^2 + x - 2$$

There is a neat way to organize your calculations that makes adding and subtracting polynomials like adding and subtracting whole numbers. We will write the polynomials in a vertical format, one over the other, with like terms lined up in columns. To produce a sum, we just add the coefficients in each of the columns. When finding a difference, we subtract the coefficients in each of the columns.

Example 4 **Find $(4x^3 + 5x^2 + 6x + 7) + (2x^3 + 3x + 4)$.**

Solution Write the polynomials in a vertical format, with like terms in columns. Since there is no x^2-term in the second polynomial we will write "$+ 0x^2$" in the polynomial to help with the alignment of like terms.

$$\begin{array}{r} 4x^3 + 5x^2 + 6x + 7 \\ \underline{+\ 2x^3 + 0x^2 + 3x + 4} \end{array}$$

Now we add the coefficients in each column, retaining the appropriate variable terms in the sum.

$$\begin{array}{r} 4x^3 + 5x^2 + 6x + 7 \\ \underline{+\ 2x^3 + 0x^2 + 3x + 4} \\ 6x^3 + 5x^2 + 9x + 11 \end{array}$$

So, $(4x^3 + 5x^2 + 6x + 7) + (2x^3 + 3x + 4) =$
$6x^3 + 5x^2 + 9x + 11$.

Point out that this process works exactly the same way for subtraction, with the bottom coefficient being subtracted from the top coefficient in each column.