Key Concepts

Combinations

Objective Teach students how to count the number of different ways one can choose a number of members from a collection, where the order is not important.

Note to the Teacher *The counting problems in this lesson require that students already understand the concept of permutations and how to count them.*

Counting Combinations

Note to the Teacher *It is a good idea to explore the concept that the number of combinations can be determined by counting the related permutations and then dividing. This exploration is best done by presenting an example involving a small number of members. Be sure to illustrate this idea carefully while presenting the following example. The example clearly shows that order is not important in combination problems, since a clean-up team of Karl and Susan is no different that a team of Susan and Karl. Students should be able to grasp this important distinction quite readily.*

Example 1 **Karl, Susan, Enrique, and Janet are members of a club. Each month, two of them have to spend an hour cleaning up the club house. How many different clean-up teams can be chosen from among the four members?**

Solution One way is make the determination is to write out each of the possibilities.

Karl and Susan, Karl and Enrique, Karl and Janet,
Susan and Enrique, Susan and Janet, Enrique and Janet

So, there are 6 possible clean-up teams.

Emphasize that in Example 1 where there are only 4 club members, writing out a list of the combinations is possible. However, if there were 10 members instead of 4, writing a list would become too time-consuming, and it is also possible that our list might be incomplete without us realizing this fact. Suggest to your students that there is a better way for determining the number of possible combinations, using an idea we have already worked with, namely *permutations*.

Remind students that the number of permutations is the number of different ways the members of a collection can be chosen *when order is important.* Refer back to Example 1. In this situation, we want to know how many different ways we can choose 2 members from among Karl, Susan, Enrique, and Janet. If the order of the two clean-up team members was important, this would be a permutation. Thinking about this in the way we did in the last lesson, the first choice can be made in 4 ways and then for each of these choices there are 3 ways to make the second choice. This yields 4×3 or 12 permutations. We can think of this situation as *4 members, chose 2.* The number of permutations of 2 things chosen from 4 things is sometimes written as $P(4, 2)$. Here is a list of the 12 permutations.

Karl and Susan, Karl and Enrique, Karl and Janet,
Susan and Karl, Susan and Enrique, Susan and Janet,
Enrique and Karl, Enrique and Susan, Enrique and Janet,
Janet and Karl, Janet and Susan, Janet and Enrique

Each one of these permutations can be thought of as a way of choosing a clean-up team, but since permutations do take order into account, each possible team is actually counted twice, since the pair of names can appear in either order. We will list each of the 6 possible clean-up teams, together with the mirror image that corresponds to it.

Karl and Susan → Karl and Susan; Susan and Karl
Karl and Enrique → Karl and Enrique; Enrique and Karl
Karl and Janet → Karl and Janet; Janet and Karl
Susan and Enrique → Susan and Enrique; Enrique and Susan
Susan and Janet → Susan and Janet; Janet and Susan
Enrique and Janet → Enrique and Janet; Janet and Enrique

Every possible team occurs exactly once in the left-hand column, and both permutations of each pair of names occurs to the right of the pairing. This means that the number of different clean-up teams can be obtained by taking the number of permutations of 4 items taken 2 at a time and dividing by the number of permutations involving only the members of one particular team. So,

$$\text{number of teams} = \frac{\text{number of permutation of 4 items taken 2 at a time}}{\text{number of permutations involving the same team}}$$

$$= \frac{12}{2} \text{ or } 6$$

Each of the six clean-up teams is an example of a **combination** taken from the collection {Karl, Susan, Enrique, Janet}. A combination is similar to a permutation, but the order of the items is *not* taken into account. That is, the pairing “Karl and Susan” is considered to be the same as the pairing “Susan and Karl.”

The Formula

Suppose n and k are numbers, with $k \leq n$. Then we will write $C(n, k)$ to represent the number of possible collections containing k members chosen from a collection containing n members. Here are three usages of this symbolism.

- $C(4, 2)$ is the number of ways of choosing 2 members from a 4-member collection without regard to order. This is the combination problem involving the clean-up teams discussed earlier. We saw that there were 6 possible clean-up crews when order was not important, so we know that $C(4, 2) = 6$.
- The number of ways of choosing a hand of 7 cards from a standard deck of 52 playing cards is denoted as $C(52, 7)$. This is a very large number.
- The number of ways of choosing 3 numbers from the collection $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is written $C(9, 3)$.

Earlier, we found that the number of different 2-person clean-up crews that can be formed when 4 persons are available is

$$C(4, 2) = \frac{\text{number of permutations of 2 members from a 4-member collection}}{\text{number of permutations of 2 members from a 2-member collection}}$$

$$= \frac{P(4, 2)}{P(2, 2)}.$$

This idea works in general.

Key Idea	$C(n, k) = \frac{P(n, k)}{P(k, k)}$

Example 2 **How many 7-card hands can be dealt from a standard deck of 52 playing cards?**

Solution In order to determine the number of 7-card hands that can be dealt from a standard deck of 52 playing cards, we can write

$$C(52, 7) = \frac{P(52, 7)}{P(7, 7)}.$$

Using our knowledge from the previous lesson, we know how to compute the number of permutations for choosing 7 items from a collection of 52 items, and also how to compute the number of permutations for choosing 7 items from a collection of 7 items.

$$P(52, 7) = \underbrace{52 \times 51 \times 50 \times 49 \times 48 \times 47 \times 46}_{7 \text{ items}} \text{ or } 674{,}274{,}182{,}400$$

$$P(7, 7) = \underbrace{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}_{7 \text{ items}} \text{ or } 5{,}040$$

Therefore, $C(52, 7) = \frac{674{,}274{,}182{,}400}{5{,}040}$ or 133,784,560.

Your students may be surprised to discover that there are more than one hundred million possible 7-card hands that can be dealt from a standard deck of playing cards.

Example 3 **In how many ways can 3 numbers be chosen from the collection {1, 2, 3, 4, 5, 6, 7, 8, 9}?**

Solution If we want to know the number of ways we can choose 3 digits from the collection {1, 2, 3, 4, 5, 6, 7, 8, 9} without regard to their order, we must compute $C(9, 3)$.

$$C(9, 3) = \frac{P(9, 3)}{P(3, 3)}$$

We know that

$$P(9, 3) = 9 \times 8 \times 7 \text{ or } 504,$$

and that

$$P(3, 3) = 3 \times 2 \times 1 \text{ or } 6.$$

So,

$$C(9, 3) = \frac{504}{6} \text{ or } 84.$$

There are 84 different combinations of 3 digits that can be chosen from the collection.