Key Concepts

Multiplying Monomials

Objective Introduce polynomials by teaching students to use the laws of exponents to multiply monomials.

Note to the Teacher *To this point, students have worked with linear expressions and equations. They now need to understand how to do algebra with higher degree expressions, involving higher powers of the variables. The key ideas in this lesson are the laws of exponents, which are used to simplify monomial expressions.*

Monomials

Begin by reviewing **exponents** with the class. Write down that if x is a variable, then x^2 denotes $x \cdot x$, x^3 denotes $x \cdot x \cdot x$, and more generally, x^n denotes

$$\underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ factors}}.$$

Definition of Monomials	A **monomial** is a number, a variable, or a product of numbers and variables. • Numbers are referred to as constants. • When there are several occurrences of the same variable, the expression is typically written in exponent form.

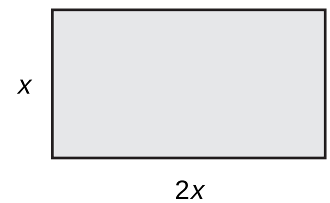
Monomials: $5x^2$, $7xy$, $-3x^3$, $4y$, z^{100}

Not Monomials: $3x + 1$, $\frac{1}{x^2 + 2}$, $\sqrt{x + 3}$

Monomials frequently occur in descriptions of practical problems.

Example 1 **The longer side of a rectangle is twice as long as the shorter side. Write an expression for the area of the rectangle.**

Solution Let x represent the length of the shorter side. Then let $2x$ represent the length of the longer side. The area of the rectangle is as follows.



$$\begin{aligned}\text{Area} &= (\text{length of longer side}) \times (\text{length of shorter side})\\ &= (2x)(x)\\ &= 2 \cdot x \cdot x\\ &= 2x^2\end{aligned}$$

So, the area is $2x^2$, which is a monomial, since it is a product of the constant 2 with two copies of the variable x.

Example 2 **At a school bake sale, each cookie costs 50 cents. Suppose that each customer buys the same number of cookies. Write a monomial representing the total sales amount.**

Solution Let c represent the number of customers.
Let x represent the number of cookies each customer buys.

Then the total number of cookies sold is

(number of customers) × (number of cookies each customer buys)

or cx.

Since each cookie sells for 50 cents, the total sales amount is $50cx$ cents. This is a monomial, since it is the product of the constant 50 and the variables c and x.

Laws of Exponents and Multiplying Monomials

When multiplying monomials, analyze the product of two powers of the same variable, for instance, $x^2 \cdot x^3$. A good way to illustrate this is by multiplying various powers of 2 together.

Example 3 **Simplify $2^2 \cdot 2^3$.**

Solution

$$\begin{aligned}2^2 \cdot 2^3 &= 4 \cdot 8\\ &= 32\\ &= 2^5\end{aligned}$$

Example 4 **Simplify $2^4 \cdot 2^5$.**

Solution $2^4 \cdot 2^5 = 16 \cdot 32$
$= 512$
$= 2^9$

In Examples 3 and 4, the exponent of the result of the multiplication is the sum of the exponents in the two factors. In Example 3, $5 = 2 + 3$, and in Example 4, $9 = 4 + 5$. Why is this true? Recall that exponents are just shorthand for a repeated product of the same number or variable. So,

$$2^2 \cdot 2^3 = \underbrace{2 \cdot 2}_{\text{2 factors}} \cdot \underbrace{2 \cdot 2 \cdot 2}_{\text{3 factors}} = \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{\text{5 factors}} = 2^5.$$

In general, when multiplying x^m and x^n, the result is as follows.

$$x^m \cdot x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{m \text{ factors}} \cdot \underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ factors}} = \underbrace{x \cdot x \cdot \ldots \cdot x}_{m + n \text{ factors}} = x^{m+n}$$

Product of Powers	When a power of x is multiplied by another power of x, the result is a power of x whose exponent is the sum of the exponents of the factors. $x^m \cdot x^n = x^{m+n}$

This is true for any number or variable x, and for any whole numbers m and n. Now use this idea to multiply monomials.

Example 5 **Simplify $3x^5 \cdot 4x^2$.**

Solution $3x^5 \cdot 4x^2 = (3 \cdot 4) \cdot (x^5 \cdot x^2)$
$= 12x^{5+2}$ or $12x^7$

Notice that the constants are grouped together and the variables are grouped together.

Example 6 **Simplify $9y^8 \cdot (-y^7)$.**

Solution $9y^8 \cdot (-y^7) = [9 \cdot (-1)] \cdot (y^8 \cdot y^7)$
$= -9y^{15}$

Note to the Teacher *Make sure the class gets plenty of opportunity to practice this kind of multiplication.*

Powers of a Monomial

What happens when a monomial is raised to a power? A good way to introduce this concept is to work again with powers of 2. First look at the squares of various powers of 2.

2^n	$(2^n)^2$
$2^1 = 2$	$4 = 2^2$
$2^2 = 4$	$16 = 2^4$
$2^3 = 8$	$64 = 2^6$
$2^4 = 16$	$256 = 2^8$
$2^5 = 32$	$1024 = 2^{10}$
$2^6 = 64$	$4096 = 2^{12}$

Notice that when a power of 2 is squared, the exponent in the result is doubled.

Example 7 **Simplify $(x^4)^3$.**

Solution To simplify the expression, write out the powers as products.

$$\begin{aligned}(x^4)^3 &= x^4 \cdot x^4 \cdot x^4 \\ &= \underbrace{x \cdot x \cdot x \cdot x}_{\text{4 factors}} \cdot \underbrace{x \cdot x \cdot x \cdot x}_{\text{4 factors}} \cdot \underbrace{x \cdot x \cdot x \cdot x}_{\text{4 factors}} \\ &= \underbrace{x \cdot x \cdot \ldots \cdot x \cdot x}_{\text{12 factors}} \\ &= x^{12}\end{aligned}$$

In this case, the exponent 4 is tripled. The exponent in the final answer is 12, since the monomial is raised to the third power. This works in general.

Power of a Power	When a power of a number or variable is raised to another power, the result is that same number or variable whose exponent is the product of the exponents. $(x^m)^n = x^{mn}$

Example 8 **Simplify $(x^8)^7$.**

Solution $(x^8)^7 = x^{8 \cdot 7}$
$= x^{56}$

Example 9 **Simplify $(15^3)^6$.**

Solution $(15^3)^6 = 15^{3 \cdot 6}$
$= 15^{18}$

Note to the Teacher *It is a good idea to work out explicitly several examples by expanding the powers into products, so that reasoning behind the key idea is reinforced.*

To explain this idea in the general case, use the following diagram.

$$(x^m)^n = \underbrace{x^m \cdot x^m \cdot \ldots \cdot x^m \cdot x^m}_{n \text{ factors}}$$

$$= \underbrace{\underbrace{x \cdot x \cdot \ldots \cdot x \cdot x}_{m \text{ factors}} \cdot \underbrace{x \cdot x \cdot \ldots \cdot x \cdot x}_{m \text{ factors}} \cdot \ldots \cdot \underbrace{x \cdot x \cdot \ldots \cdot x \cdot x}_{m \text{ factors}}}_{n \text{ factors}}$$

Point out that there are n groups of factors, each of which is itself a group of m factors, so that there are mn factors all together. So, the result is x^{mn}.

Students have now learned how to find powers of powers. Since monomials are generally products of powers, introduce how to find powers of products. This will be helpful in finding powers of monomials. Write $(a \cdot b)^2$ on the chalkboard, and ask students how they think the expression should be simplified. The following solution can be given as an explanation.

$$\begin{aligned}(a \cdot b)^2 &= (a \cdot b) \cdot (a \cdot b)\\ &= a \cdot b \cdot a \cdot b\\ &= a \cdot a \cdot b \cdot b\\ &= a^2 \cdot b^2\end{aligned}$$

This idea is shown below using numbers.

$$6^3 = 216$$

$$\begin{aligned}6^3 &= (2 \cdot 3)^3\\ &= 2^3 \cdot 3^3\\ &= 8 \cdot 27\\ &= 216\end{aligned}$$

Power of a Product	A product raised to a power is the product of the factors raised to the given power. $(a \cdot b)^m = a^m \cdot b^m$

A good way to show this is to use the following diagram.

$$(a \cdot b)^m = \underbrace{(a \cdot b) \cdot (a \cdot b) \cdot \ldots \cdot (a \cdot b)}_{m \text{ factors}} = \underbrace{a \cdot a \cdot \ldots \cdot a}_{m \text{ factors}} \cdot \underbrace{b \cdot b \cdot \ldots \cdot b}_{m \text{ factors}} = a^m b^m$$

Emphasize that in the middle expression, there are exactly *m* factors each of *a* and *b*.

Now combine these two ideas into one.

Power of a Monomial	For any whole numbers *m*, *n*, and *p*, $(a^m \cdot b^n)^p = a^{mp}b^{np}$.

Now students can evaluate powers and products of any monomials. Present several exercises and ask students to practice the technique.

Exercises

Simplify each expression.

1. $(2xb^2)^3$ $\mathbf{8x^3b^6}$

2. $(3xy)^5(x^2y)^4$ $\mathbf{243x^{13}y^9}$

3. $(a^3b^2)^2 \cdot b^7$ $\mathbf{a^6b^{11}}$

4. $100y^2z^2 \cdot (3z)^5$ $\mathbf{24{,}300y^2z^7}$