# Key Concepts

## Graphing Inequalities in Two Variables

**Objective** Teach students to graph the solution sets of inequalities in two variables.

**Note to the Teacher** *It is important that the class understand that solving an inequality in two variables means describing the solution set for the inequality, and that graphing it means specifying a whole region, rather than a line or a curve. The solution procedure is similar to that for linear equations, involving the Addition and Multiplication Properties of Inequalities. That too should be emphasized, since it connects these new ideas with ideas that are already familiar to students.*

### Linear Inequalities

A **linear inequality** is an expression similar to a linear equation, except that it has an inequality symbol rather than an equals sign.

| Linear Inequalities | Not Linear Inequalities |
|---|---|
| $2x + 3y \leq 7 + 5x$ | $x^2 + 5 \geq y$ |
| $x + 5 \geq 2y - 5$ | $xy > 7$ |
| $y < 5$ | $y = x - 4$ |

### Solution Sets to Linear Inequalities

Begin with an inequality in two variables, say $3y + 5 \leq 2x - 1$. Then the solution set for the inequality is the collection of all ordered pairs $(x, y)$ for which the inequality holds true. For example, the ordered pair (7, 0) is in the solution set because substituting 7 for $x$ and 0 for $y$ makes the inequality true.

$$3(0) + 5 \leq 2(7) - 1 \quad \textit{Replace } (x, y) \textit{ with } (7, 0).$$

$$5 \leq 13 \quad ✓$$

$5 \leq 13$ is a valid inequality.

On the other hand, the ordered pair (2, 4) is not in the solution set because substituting 2 and 4 for $x$ and $y$, respectively, makes the inequality false.

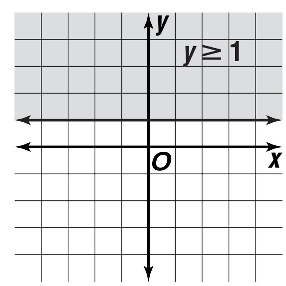
$$3(4) + 5 \leq 2(2) - 1 \qquad \textit{Replace } (x, y) \textit{ with } (2, 4).$$
$$17 \not\leq 3$$

$17 \leq 3$ is not a valid inequality.

**Note to the Teacher** *Have students practice determining whether a particular point is in the solution set. Give them various points, and ask them to determine if they are in the solution set for the given inequality.*

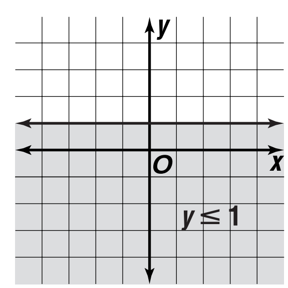
## Graphing Solution Sets for Inequalities

First, take a very simple inequality, $y \geq 1$. The solution set consists of all points whose $y$-coordinate is greater than or equal to 1. These points are contained in the shaded region in the graph at the right.




This kind of region is called a **half-plane** because it is one of two parts of the plane into which a boundary line divides it. In this case, the region consists of all those points that lie on and above the line $y = 1$.

Another example is $y \leq 1$. The "greater than or equal to" is changed to "less than or equal to." The solution set for this inequality is shown at the right.

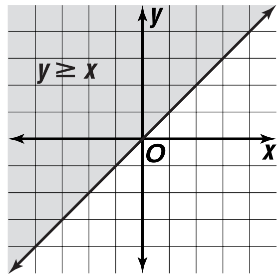


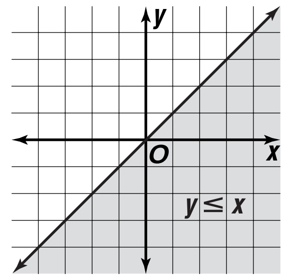

It is also a half-plane. In this case, the solution set consists of all points in the half-plane including and below the line $y = 1$.

In both cases, the equation of the boundary line is found by replacing the inequality symbol with an equals sign.

Consider $y \geq x$. The equation of the boundary line is $y = x$, which is found by replacing the inequality symbol with an equals sign. The solution set for the inequality is the half-plane including and above the line. In the same way, the graph of the solution set for $y \leq x$ is the half-plane including and below this same line.

The solution sets for both inequalities are shown below.



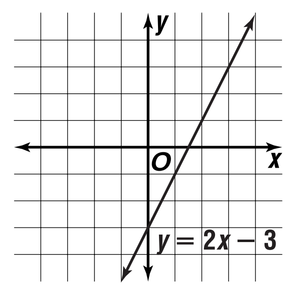




The following general key idea is always true.

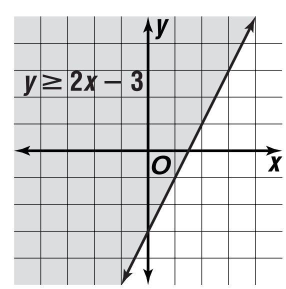
| **Key Idea** | For any linear inequality, if the inequality symbol is replaced with an equals sign, the result is a line that divides the plane into two half-planes. The solution set for the inequality is one of these half-planes. |
|---|---|

**Example 1** **Graph $y \geq 2x - 3$.**

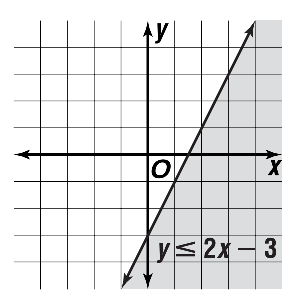
**Solution** First, replace the inequality symbol in $y \geq 2x - 3$ with an equals sign, in this case $y = 2x - 3$. Graph the line.




Now, since the inequality states that the $y$-coordinate is greater than or equal to the linear expression in $x$, the solution set for the inequality is the set of points above this line. This is shown in the shaded region at the right.




If the inequality symbol were reversed and the inequality was $y \leq 2x - 3$, the solution set would be the set of points below this line, as shown at the right.



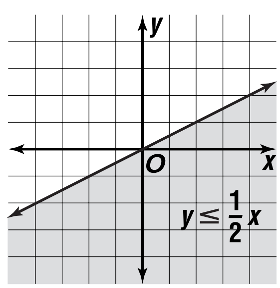
| Key Idea | • If a linear inequality sets *y* greater than or equal to the linear expression in *x*, then the solution set is the set of points *above* the boundary line.<br>• If a linear inequality sets *y* less than or equal to the linear expression in *x*, then the solution set is the set of points *below* the boundary line. |
|---|---|

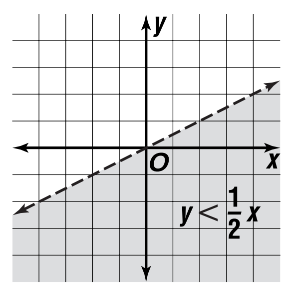
Using this key idea, the solution set for $y \geq 5x - 7$ is the set of points above the line $y = 5x - 7$. The solution set for $y \leq x - 1$ is the set of points below the line $y = x - 1$.

So far, only inequalities containing $\leq$ and $\geq$ have been graphed. Inequalities with the symbols $<$ and $>$ are just as easy to graph.

| Key Idea | • When the inequality symbol is $\leq$ or $\geq$, draw a solid line on the boundary of the half-plane to indicate that the boundary line is included.<br>• When the inequality symbol is $<$ or $>$, draw a dashed line on the boundary of the half-plane to indicate that the boundary line is not included. |
|---|---|

This key idea is illustrated by the graphs shown below.




## Solving Linear Inequalities

Solving a linear inequality means writing it in a form so that only *y* appears on the left-hand side, and only *x* and constants appear on the right side. This is similar to solving an equation for *y*. It is easy to graph the solution sets for linear inequalities. Simply graph the corresponding boundary line and then shade the region either above or below the line.

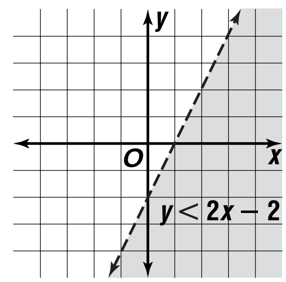
Solving linear inequalities can be done by using the Addition and Multiplication Properties of Inequalities, just as the corresponding properties of equations are used to solve linear equations. This is best illustrated by examples.

**Example 2** **Solve $y + 7 < 2x + 5$.**

**Solution** First, rewrite the inequality so that $y$ is on the left-hand side by itself.

$$y + 7 < 2x + 5$$
$$y < 2x - 2 \qquad \textit{Subtract 7 from each side.}$$

So, the solution set is the set of all points that lie below the line $y = 2x - 2$, as shown in the graph at the right.



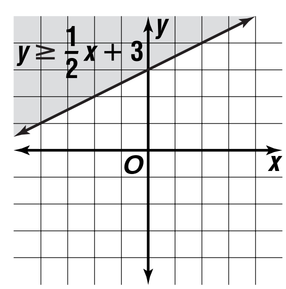

Notice that there is a dashed line on the boundary, since the inequality symbol is $<$, not $\leq$.

**Example 3** **Solve $2y \geq x + 6$.**

**Solution** To solve the inequality, first divide each side by 2 to isolate the $y$.

$$2y \geq x + 6$$
$$y \geq \frac{1}{2}x + 3 \qquad \textit{Divide each side by 2.}$$

The solution set is the set of points on and above the line $y = \frac{1}{2}x + 3$, as shown in the graph at the right.



**Example 4** **Solve $3 - y \leq x + 5$.**

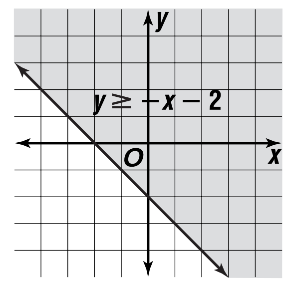
**Solution** First, use the properties of inequalities to isolate the $y$.

$$3 - y \leq x + 5$$

$$-y \leq x + 2 \qquad \textit{Subtract 3 from each side.}$$

$$y \geq -x - 2 \qquad \textit{Multiply each side by } -1\textit{. Reverse the inequality symbol.}$$

The graph of $y \geq -x - 2$ is shown at the right. Notice that a solid line is on the boundary of the region.




Recall that the Multiplication Property of Inequalities states that multiplying each side of an inequality by the same positive number produces an equivalent inequality. However, when multiplying each side by the same negative number, the inequality symbol must be reversed to get an equivalent inequality.