Key Concepts

Exponential Functions

Objective Guide students in becoming familiar with exponential functions: their basic properties, their graphs, and their applications.

Note to the Teacher *In this lesson, students will be introduced to exponential functions: how to compute with them, and how to use them. Begin by reviewing properties of exponents found in Lesson 9-1 in the Student Edition.*

Graphs of Exponential Functions

Begin by telling students that our first goal is to define functions that involve exponents. For example, write a table of powers of 2 on the chalkboard, like the one shown at the right.

n	2^n
−4	$\frac{1}{16} = 0.0625$
−3	$\frac{1}{8} = 0.125$
−2	$\frac{1}{4} = 0.25$
−1	$\frac{1}{2} = 0.5$
0	1
1	2
2	4
3	8
4	16
5	32

Observe that if one takes large powers of 2, the values get very large. If one takes negative powers of 2, the values are fractions between 0 and 1. The larger the absolute value of the negative power, the smaller the fraction. In particular, observe that all the values of 2^n are positive, no matter what n is. The goal of this lesson is to consider 2^n as a function of n, and more generally, the function $y = 2^x$, or more generally yet, functions of the form $y = a^x$, where a is a positive number. These are called **exponential functions**.

Definition of Exponential Function	An exponential function is a function that can be described by an equation of the form $y = a^x$, where $a > 0$, and $a \neq 1$.

Explain that $a \neq 1$ because the value of $y = 1^x$ is always 1, and $y = 1$ is a linear function.

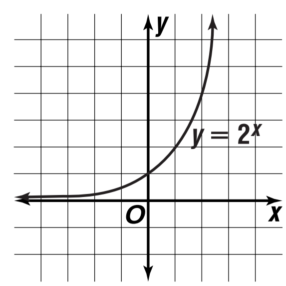
To help students better understand exponential functions, first review some of the properties of exponents that they studied in Lessons 9-1 and 9-2.

Product of Powers	**Quotient of Powers**	**Negative Exponents**
$x^m \cdot x^n = x^{m+n}$	$\frac{x^m}{x^n} = x^{m-n}$	$x^{-n} = + \frac{1}{x^n}$

These properties hold when x is any positive number, and m and n are any numbers.

Use the table of powers of 2 to graph $y = 2^x$. Have students study the graph to see the following properties of this function.

- The value of $y = 2^x$ is always positive, even when x is negative. In particular, $2^0 = 1$.
- As x gets large, $y = 2^x$ gets very large. So as x approaches infinity, y approaches infinity at a very fast rate. This is referred to as **exponential growth**.
- As x approaches negative infinity, $y = 2^x$ approaches zero.

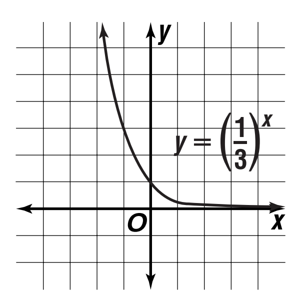


These properties are also true of the function $y = a^x$ for $a > 1$.

1. The value of $y = a^x$ is always positive. In particular, $a^0 = 1$.
2. As x gets large (approaching infinity), then $y = a^x$ approaches infinity.
3. As x approaches negative infinity, $y = a^x$ approaches zero.

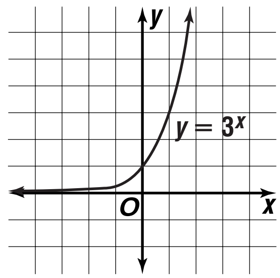
When a is a positive number less than 1, the graph of $y = a^x$ has a different quality. As we can see from the graph of $y = \left(\frac{1}{3}\right)^x$ shown below, for a positive number a less than 1, the function $y = a^x$ has the following properties.

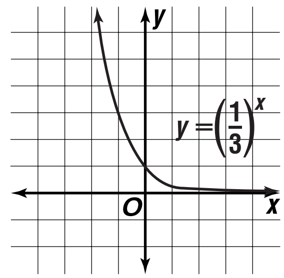
1. The value of $y = a^x$ is always positive. In particular, $a^0 = 1$.
2. As x gets large (approaching infinity), then $y = a^x$ approaches zero.
3. As x approaches negative infinity, $y = a^x$ approaches infinity.



Why are the properties different when $a < 1$ (compared to $a > 1$)? Consider $y = \left(\frac{1}{3}\right)^x$.

Recall that $\frac{1}{3} = 3^{-1}$ so that $y = \left(\frac{1}{3}\right)^x = (3^{-1})^x = 3^{-x}$. Thus, the graph of $y = \left(\frac{1}{3}\right)^x = 3^{-x}$ looks like the graph of $y = 3^x$ except that x and $-x$ are reversed. So as x approaches positive infinity, $-x$ approaches negative infinity. We can see this phenomenon by graphing $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$.





Note to the Teacher *Now that students have some familiarity with the graphs of exponential functions, the next step is for them to become comfortable solving equations involving exponential functions. The main principle involved in solving exponential equations is the following property, which you should write on the chalkboard.*

Solving Exponential Equations

Property of Equality for Exponential Functions	Suppose a is a positive number other than 1. Then $a^{x_1} = a^{x_2}$ if and only if $x_1 = x_2$.

Now do an example.

Example 1 **Solve $5^{3x+4} = 5^x$.**

Solution Let $x_1 = 3x + 4$ and let $x_2 = x$. Then by the property of equality, if $5^{3x+4} = 5^x$, then $3x + 4 = x$. Now solve this equation.

$$3x + 4 = x$$

$2x + 4 = 0$ *Subtract x from each side.*

$2x = -4$ *Subtract 4 from each side.*

$x = -2$ *Divide each side by 2.*

You can check this solution by substituting -2 for x in the original equation.

Example 2 **Solve $2^{x+1} = 8^{3x}$.**

Solution In this case, the two quantities do not have the same base, or value of *a*. In order to use the property of equality, rewrite one of the quantities so that they will both have the same base. Notice that $8 = 2^3$. So, we can rewrite 8^{3x} as $(2^3)^{3x} = 2^{9x}$. The equation we need to solve becomes

$$2^{x+1} = 2^{9x}.$$

By the Property of Equality for Exponential Functions, this equation is true if and only if $x + 1 = 9x$. Now solve this equation.

$$x + 1 = 9x$$

$$1 = 8x \quad \textit{Subtract } x \textit{ from each side.}$$

$$\frac{1}{8} = x \quad \textit{Divide each side by 8.}$$

Example 3 **(from p. 641 of the Student Edition) Mitosis is a process of cell reproduction in which one cell divides into two identical cells. *E. coli* is a fast-growing bacteria that is often responsible for food poisoning in uncooked meat. It can reproduce itself in 15 minutes. If you begin with 100 *E. coli* bacteria, how many bacteria will there be in 1 hour?**

Solution We start with 100 bacteria, and after 15 minutes there are 2×100 bacteria, and after 30 minutes there are $2 \times 2 \times 100$ bacteria, and after 45 minutes there are $2 \times 2 \times 2 \times 100$ bacteria, and after 1 hour there are $2 \times 2 \times 2 \times 2 \times 100$ or 1600 bacteria.

Notice that the population of bacteria can then be described as a function of time. Let *t* represent the time that has elapsed measured in units of 15 minutes. In other words when $t = 1$, 15 minutes have elapsed, when $t = 2$, 30 minutes have elapsed, and so on. Then the population as a function of *t* is given by

$$y = 2^t \times 100.$$

One hour is represented by $t = 4$, so the population after 1 hour is $y = 2^4 \times 100$ or 1600 bacteria. Now that we have the general formula, we can compute the population of *E. coli* at any time. For example, the population after $2\frac{1}{2}$ hours can be computed by first noticing that $2\frac{1}{2}$ hours consists of 10 units of 15 minutes. So, substitute $t = 10$ into the formula. Namely, the population of *E. coli* after $2\frac{1}{2}$ hours is given by

$$y = 2^{10} \times 100 \text{ or } 102{,}400 \text{ bacteria.}$$

End of Lesson