Key Concepts

Solving Quadratic Equations by Completing the Square

Objective Teach students the technique of completing the square, and how to use this technique to solve quadratic equations and to verify the Quadratic Formula.

Note to the Teacher *Completing the square is an important technique in working with quadratic equations. Explain to your students that the technique is motivated by the simplest example of a quadratic equation, when the square of a variable is equal to a positive number.*

Taking the Square Root of Each Side

Example 1 **Solve $x^2 = 9$.**

Solution To solve this equation, simply take the square root of each side.

$$x^2 = 9$$
$$x = \pm\sqrt{9}$$
$$x = \pm 3$$

You can also use this method to solve a quadratic equation in which the square of an expression of the form $(x - c)$ is equal to a positive number.

Example 2 **Solve $(x - 3)^2 = 16$.**

Solution Again, take the square root of each side.

$$(x - 3)^2 = 16$$
$$x - 3 = \pm\sqrt{16}$$
$$x - 3 = \pm 4$$
$$x - 3 = 4 \quad \text{or} \quad x - 3 = -4$$
$$x = 7 \qquad\qquad x = -1$$

These examples show how to solve quadratic equations of the form $(x + k)^2 = d$ or $(x + k)^2 - d = 0$. This is called *completed square form.* Quadratic equations that are not in this form can be rewritten using the following technique.

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Completing the Square

The technique of completing the square uses algebra to put any quadratic equation in completed square form. To understand how to write a quadratic expression of the form $x^2 + bx$ as a perfect square, work backwards.

$$(x + k)^2 = x^2 + 2kx + k^2$$

Point out to your students that the coefficient of x in this expression is $2k$. So, to write $x^2 + bx$ as a perfect square trinomial $x^2 + bx + c$, let $b = 2k$, or $k = \frac{b}{2}$.

Example 3 **Write $x^2 + 12x$ as a perfect square trinomial $x^2 + 12x + c$, where c is a constant.**

Solution First, find $\frac{b}{2}$. In this quadratic expression, $b = 12$. $\quad \frac{b}{2} = \frac{12}{2}$ or 6

Next, square this result. $\quad 6^2 = 36$

Finally, add the result to the original expression. $\quad x^2 + 12x + 36$

This is a perfect square trinomial since $x^2 + 12x + 36 = (x + 6)^2$.

Stress to your students that completing the square is very helpful in solving quadratic equations. Do several examples in class.

Example 4 **Solve $x^2 + 6x + 3 = 0$.**

Solution Since $x^2 + 6x + 3$ is not a perfect square, subtract 3 from each side and then complete the square of the quadratic expression.

$$x^2 + 6x + 3 = 0$$

$$x^2 + 6x = -3 \quad \textit{Subtract 3 from each side.}$$

First, find $\frac{b}{2}$. In the expression $x^2 + 6x$, $b = 6$. $\quad \frac{b}{2} = \frac{6}{2}$ or 3

Next, square this result. $\quad 3^2 = 9$

Finally, add the result to each side of the equation. $\quad x^2 + 6x + 9 = -3 + 9$

Now simplify.

$$x^2 + 6x + 9 = -3 + 9$$

$$(x + 3)^2 = 6 \quad \textit{Factor } x^2 + 6x + 9.$$

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This equation is in comMpleted square form. Now solve as in Example 2, by taking the square root of each side.

$$(x + 3)^2 = 6$$
$$x + 3 = \pm\sqrt{6}$$
$$x + 3 = \sqrt{6} \quad \text{or} \quad x + 3 = -\sqrt{6}$$
$$x = \sqrt{6} - 3 \qquad x = -\sqrt{6} - 3$$

The method of completing the square will also let you know when there are no solutions to a quadratic equation, as shown in the following example.

Example 5 **Solve $x^2 - x + 2 = 0$.**

Solution Since the expression is not a perfect square, begin by subtracting 2 from each side of the equation to get $x^2 - x = -2$. Then complete the square of the expression $x^2 - x$.

First, find $\frac{b}{2}$. In this expression, $b = -1$. $\frac{b}{2} = \frac{-1}{2}$ or $-\frac{1}{2}$

Next, square this result. $\left(-\frac{1}{2}\right)^2 = \frac{1}{4}$

Finally, add the result to each side of the equation. $x^2 - x + \frac{1}{4} = -2 + \frac{1}{4}$

Now simplify.

$$x^2 - x + \frac{1}{4} = -2 + \frac{1}{4}$$

$$\left(x - \frac{1}{2}\right)^2 = -\frac{7}{4} \qquad \textit{Factor } x^2 - x + \frac{1}{4}.$$

The expression $\left(x - \frac{1}{2}\right)^2$ is always greater than or equal to zero because the square of any number is greater than or equal to zero. However, the right-hand side of this equation, $-\frac{7}{4}$ is negative. Therefore, there can be no solutions to this equation. Thus, the original equation $x^2 - x + 2 = 0$ has no real solution.

Note to the Teacher *As a review, ask your students what it means graphically when there are no real solutions to a quadratic equation $ax^2 + bx + c = 0$.* **It means that the graph of the quadratic function $y = ax^2 + bx + c$ (a parabola) does not intersect the x-axis.**

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Proving the Quadratic Formula

We have seen that the method of completing the square is very useful in solving quadratic equations. It is also important in proving the Quadratic Formula. This is probably the most important formula in this course, and its proof is very important. The proof is carried out in a step-by-step manner on page 745 of the Student Edition. Go through that proof slowly and carefully in class.

End of Lesson

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